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Homogenization of Semi-linear Optimal Control Problems on Oscillating Domains with Matrix Coefficients

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Abstract
In this article, we study the homogenization of optimal control problems subject to second-order semi-linear elliptic PDEs with matrix coefficients in two different types of oscillating domains: a circular domain and a domain with general low-dimensional oscillations. The cost functionals considered are of general energy type with oscillating matrix coefficients, and the coefficient matrix in the cost functional is allowed to differ from the coefficient matrix in the constrained PDE. We prove well-defined limit problems for both domains and obtain explicit forms for the limiting coefficient matrices of the cost functionals and constrained PDEs. As expected, the coefficient matrix of the limit cost functional is a combination of the original cost functional's and constrained PDE's coefficient matrices.

Keywords Homogenization · Periodic unfolding · Oscillating boundary · Circular oscillating domain

Mathematics Subject Classification 49J20 · 80M35 · 35B27

1 Introduction

In this article, we plan to study the homogenization of a semi-linear elliptic PDE in a circular oscillating domain of the form

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$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_\varepsilon) + k(u_\varepsilon) + u_\varepsilon = f & \text{in } \mathcal{O}_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon = 0 & \text{on } \partial\mathcal{O}_\varepsilon. \end{cases}$$

Here $\mathcal{O}_\varepsilon$ is the oscillating circular domain to be defined in Subsect. 2.1. The limit is quite interesting (see (5)), and the main ingredient in the analysis is the Browder–Minty method to deal with the non-linearity k together with the method of unfolding. In addition to the homogenization, we also establish corrector results, and we use these corrector results to study an associated optimal control problem with a cost functional of the form

$$J_\varepsilon(u, \theta) = \frac{1}{2} \int_{\mathcal{O}_\varepsilon} B^\varepsilon \nabla u \nabla u + \frac{\beta}{2} \int_{\mathcal{O}_\varepsilon} \chi_\omega |\theta|^2.$$

Here note that the matrix B^ε is different from the matrix A^ε of the system discussed above. In the second part of the article, we study homogenization of semi-linear PDE on n dimensional oscillating boundary domains with oscillations in m directions, $1 \leq m \leq n-1$. For example (see Fig. 4), when $n = 3$, oscillations can be of pillar type ($m = 2$) or slab type ($m = 1$). Finally, we study the optimal control problem also in this domain with an energy type cost functional.

Mathematical findings from the field of optimal control posed for domains with highly oscillating interfaces and boundaries can be used to bring insights into a large class of complex mathematical models describing a large variety of physical phenomena. Typical examples are flows through complex domains and materials with highly functional interfaces. The list includes lubricating flows with rough contacts, propagation of electromagnetic waves through the rough interface, flows in channels with rough boundaries, airflow through compression systems in turbo-machines such as jet engines, etc. The last scenario can be modeled by the viscous Moore–Greitzer equation directly derived from scaled Navier–Stokes equations. Materials with oscillating boundaries that have a designed macroscopic functionality are used in industrial applications like microstrip radiators and nanotechnologies, fractal type constructions, etc.; see e.g. [35, 39, 40]. It is not possible to give exhaustive literature here. However, we present the relevant literature in view of the problems under study.

In the context of optimal control, homogenization can be used to simplify the optimization problem by replacing the original, periodically structured system with an equivalent, homogenized system. This can be useful when the original system is too complex to be analyzed directly or when the periodicity of the system allows for significant computational simplification. In this article, as discussed above, we examine the homogenization of semi-linear optimal control problems in oscillating boundary domains where the non-linearity appears in the constrained partial differential equation (PDE). The considered problems represent a significant generalization of the results presented in previous articles [6, 45]. In [6], the authors considered an optimal control problem with a quadratic cost functional in an oscillating domain which is constrained by a second-order semi-linear elliptic PDE with a Laplacian as the principal part. In [45], the authors investigated an optimal control problem with an energy-type cost functional subject to a general second-order linear elliptic PDE with oscillating coefficients in oscillating domains with a curved interface.

There is a considerable body of literature on the homogenization of oscillating boundary domains. References such as [2, 7, 16, 30, 32, 34] and their respective sources provide extensive coverage in this area. Regarding the homogenization of optimal control problems in oscillating domains, studies employing the periodic unfolding operator to characterize the optimal control play a crucial role in the analysis. Notable references include [1, 4, 5, 45, 46, 49, 50]. For further literature on the homogenization of optimal control problems, one can refer to [24, 28, 29, 44, 47, 48] and the references therein. Significant research has also been conducted in the field of homogenization of controllability problems. References such as [19, 20, 25–27] and their respective sources focus on the homogenization of approximate controllability and exact controllability. In the recent article [27], a general approach is provided for obtaining approximate controls for parabolic problems using periodic approximations.

Regarding the homogenization of non-linear problems, a lot of literature is available. In [31], authors provide an analysis of the asymptotic behavior of a monotone-type operator with nonlinear Signorini boundary conditions. Additionally, the homogenization of a nonlinear monotone problem in a locally periodic domain using the unfolding method is studied in [8]. Another approach, the asymptotic expansion method, is employed in [38] to investigate the homogenization of a nonlinear parabolic problem. Regarding the homogenization of the semilinear optimal control problem, one interesting work is [22], where the authors focused on the homogenization of semi-linear optimal control and controllability problems in perforated domains. In the present article, the analysis became different and interesting due to the type of oscillations (refer to Figs. 1 and 4) and the nature of the cost functionals being considered. For further reading on homogenization of non-linear problems, refer [13, 14, 33, 37]. The literature on the homogenization of non-linear optimal control problems is very limited.

The main techniques used in this analysis are the unfolding operator and the monotone operator technique. The periodic unfolding method, first introduced in [21], is a powerful tool in the theory of homogenization. In [23], a modified version of this method was used to homogenize problems in pillar-type oscillating domains. The unfolding operator was further generalized to general periodic oscillating domains in [3]. The unfolding operator is also very effectively used in the context of multi-scale analysis in domains with small oscillating boundaries that is to say when homogenization and dimension reduction may take place simultaneously. An adaptation of unfolding for the thin/small oscillating was introduced in [17] to study the asymptotic behavior of viscous fluid flow through a slightly rough wall. Further, in [9], a modified version of the unfolding operator was introduced for thin porous media. Recently, several modified versions of unfolding operators are introduced depending on the nature of oscillation in the thin domain; see [10–12, 42, 43] and references therein. For more information on unfolding operators, see [18] and its references. The monotone operator technique in homogenization can be found in [6, 36, 41] and their references.

The layout of the article is as follows. Major contributions of this article are the Theorems 1, 2, 5, 6, 7, and 9. Our goal is to homogenize the optimal control problem, which requires homogenization and corrector results for the associated semi-linear PDE. Theorems 1 and 2 prove the homogenization and corrector results in circular oscillating domains. Using these theorems, we prove the homogenization of the associated optimal control problem in Theorem 5. Theorems 6 and 7 establish the homogenization and corrector results for the semi-linear PDE in $(n-m)$-dimensional oscillating domains in rectangular coordinates. Using these theorems, we obtain the homogenization results for the optimal control problem in Theorem 9 for $(n-m)$-dimensional oscillating domains.

The rest of this work is organized as follows. In Sect. 2, we homogenize the considered PDE and its associated optimal control problem in the circular oscillating domain. It is divided into several subsections. In Subsect. 2.1, we describe the domain and provide the necessary assumptions. The main tool for this section, the unfolding operator, is introduced in Subsects. 2.2 and 2.3. The main homogenization and corrector results for the considered semi-linear PDE without control are presented in Subsect. 2.4. The homogenization of the optimal control problem associated with the semi-linear PDE in the circular domain is studied in Subsect. 2.5.

In Sect. 3, we consider the homogenization of the considered PDE and its associated optimal control problem in the rectangular oscillating domain. It is divided into several subsections. In Subsect. 3.1, we describe the domain and provide the necessary assumptions. The main tool for this section, the unfolding operator, is introduced in Subsect. 3.2. The main homogenization and corrector results for the considered semi-linear PDE without control are presented in Subsect. 3.3. The homogenization of the optimal control problem associated with the semi-linear PDE in the rectangular domain is studied in Subsect. 3.4.

2 Homogenization in Circular Oscillating Domain

In this section, we investigate the homogenization of a semi-linear optimal problem in a two-dimensional domain $\mathcal{O}_\varepsilon$ that exhibits circular oscillations (as shown in Fig. 1). The homogenization of such domains has been extensively examined in prior studies (see [3, 4, 51, 52] for references).

2.1 Domain Description

Let $0 < r_0 < r_1 < r_2$ be real numbers, $\varepsilon = \frac{1}{n}, n \in \mathbb{N}$. Let Λ be a connected open subset of $\mathbb{R}^2$, which is contained in the annulus $\mathcal{O}^+ = \{(r,\theta) : r_0 < r < r_1\}$ with Lipschitz boundary as shown in Fig. 2 which is our reference cell. Now define

$$\mathcal{O}_\varepsilon^+ = \left\{(r,\theta) \in \mathcal{O}^+ : \left(r, \left\{\frac{\theta}{\varepsilon}\right\}_{2\pi}\right) \in \Lambda\right\}, \quad \mathcal{O}^- = \{(r,\theta) : r_1 < r < r_2\},$$
$$\mathcal{O}_\varepsilon = \operatorname{int}\left(\overline{\mathcal{O}_\varepsilon^+ \cup \mathcal{O}^-}\right) \quad \text{and} \quad \mathcal{O} = \operatorname{int}\left(\overline{\mathcal{O}^+ \cup \mathcal{O}^-}\right),$$

where $\mathcal{O}_\varepsilon^+$ is the inner oscillating part, $\mathcal{O}^-$ is the outer fixed part, $\mathcal{O}_\varepsilon$ is the oscillating domain and $\mathcal{O}$ is the limit domain (see Fig. 3). It is important to note that as per the definition of $\mathcal{O}_\varepsilon$, the inner part $\mathcal{O}_\varepsilon^+$ exhibits periodic oscillations. These oscillations

Fig. 1 Circular domain $\mathcal{O}_\varepsilon$

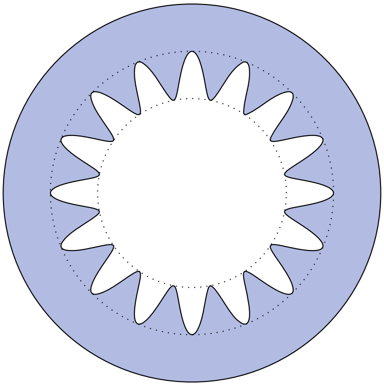


Fig. 2 Reference cell Λ

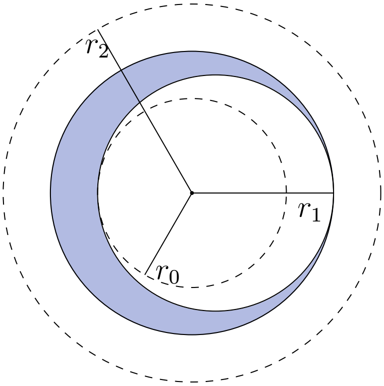
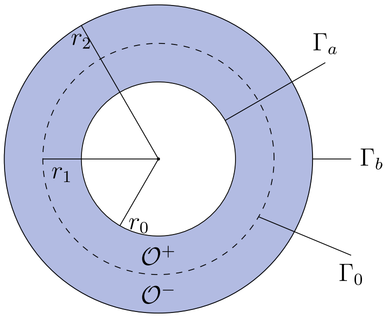


Fig. 3 Limit domain $\mathcal{O}$



involve a periodic arrangement of the reference cell Λ, which is scaled by ε in the θ variable and arranged in the θ direction with a period of $2\pi\varepsilon$. Also Γ_a, Γ_b are inner and outer boundaries of $\mathcal{O}$ and Γ_0 is the interface. Here $\left\{\frac{\theta}{\varepsilon}\right\}_{2\pi} = \frac{\theta}{\varepsilon} - \left[\frac{\theta}{2\pi\varepsilon}\right]2\pi$, where $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts. For $r \in (r_0, r_1)$, define

$$Y(r) = \{\theta \in [0, 2\pi] : (r, \theta) \in \Lambda\}.$$

We will make the following assumptions about the reference cell Λ:

1. The set $Y(r)$ is connected for all $r \in (r_0, r_1)$.
2. There exists $\rho > 0$ such that $0 < \rho \leq |Y(r)| < 2\pi$ for all $r \in (r_0, r_1)$ where $|Y(r)|$ denotes the Lebesgue measure on $\mathbb{R}$.

For completeness, we will state the definition of the polar unfolding operator for $\mathcal{O}_\varepsilon$ and list its properties.

2.2 Polar Unfolding Operator

Since the oscillations in $\mathcal{O}_\varepsilon$ occur in an angular direction, we will use unfolding operators in polar coordinates to analyze them. Here, we will provide the definition of the unfolding operator for $\mathcal{O}$ and its properties, without providing proof (for proof, see [3]).

First, we will define the unfolded domain $\mathcal{O}_U$ in which the unfolded function will be defined. The unfolded domain $\mathcal{O}_U$ is defined as follows:

$$\mathcal{O}_U = \{(r, \theta, \tau) \mid \theta \in (0, 2\pi), r \in (r_0, r_1), \tau \in Y(r)\}.$$

Let $\mathcal{G} = \{(r, \tau) \mid r \in (r_0, r_1), \tau \in Y(r)\}$, then, we can write, $\mathcal{O}_U = (0, 2\pi) \times \mathcal{G}$. Let $\phi^\varepsilon : \mathcal{O}_U \to \mathcal{O}_\varepsilon^+$ be defined as $\phi^\varepsilon(r, \theta, \tau) = \left(r, \varepsilon \left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right)$. The ε-unfolding of a function $u : \mathcal{O}_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \mathcal{O}_U \to \mathbb{R}$. The operator which maps every function $u : \mathcal{O}_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. Let the unfolding operator be denoted by T^ε, that is,

$$T^\varepsilon : \{u : \mathcal{O}_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon(u) : \mathcal{O}_U \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon(u)(r, \theta, \tau) = u\left(r, \varepsilon \left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right),$$

where $\left[\frac{\theta}{\varepsilon}\right]_{2\pi}$ denotes the integer part of $\frac{\theta}{2\pi\varepsilon}$.

If $U \subset \mathbb{R}^2$ containing $\mathcal{O}_\varepsilon^+$ and u is a real valued function on U, $T^\varepsilon(u)$ will mean, T^ε acting on the restriction of u to $\mathcal{O}_\varepsilon^+$. Some important properties of the circular unfolding operator are stated below. For each $\varepsilon > 0$,

1. T^ε is linear. Further, if $u,\ v : \mathcal{O}_\varepsilon^+ \to \mathbb{R}$, then, $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.
2. Let $u \in L^1(\mathcal{O}_\varepsilon^+)$. then,

$$\int_{\mathcal{O}_U} T^\varepsilon(u) = 2\pi \int_{\mathcal{O}_\varepsilon^+} u.$$

3. Let $u \in L^2(\mathcal{O}_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(\mathcal{O}_U)$ and $\|T^\varepsilon u\|_{L^2(\mathcal{O}_U)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)}$.
4. Let $u, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta} \in L^2(\mathcal{O}^+)$, Then, $T^\varepsilon u$, $\frac{\partial}{\partial r}T^\varepsilon u$ and $\frac{\partial}{\partial \tau}T^\varepsilon u \in L^2(\mathcal{O}_U)$. Moreover,

$$\frac{\partial}{\partial r}T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial r} \quad \text{and} \quad \frac{\partial}{\partial \tau}T^\varepsilon u = \varepsilon T^\varepsilon \frac{\partial u}{\partial \theta}.$$

5. Let $u \in L^2(\mathcal{O}_\varepsilon^+)$. Then, $T^\varepsilon u \to u$ strongly in $L^2(\mathcal{O}_U)$. More generally, let $u_\varepsilon \to u$ strongly in $L^2(\mathcal{O}^+)$. Then, $T^\varepsilon u_\varepsilon \to u$ strongly in $L^2(\mathcal{O}_U)$.
6. Let, for every ε, $u_\varepsilon \in L^2(\mathcal{O}_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\mathcal{O}_U)$. Then,

$$\widetilde{u}_\varepsilon \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} u(r,\theta,\tau)d\tau \text{ weakly in } L^2(\mathcal{O}^+),$$

where $\widetilde{u}_\varepsilon$ denotes the extension by 0 of u_ε to $\mathcal{O}^+$.
7. Let, for every $\varepsilon > 0$, $u_\varepsilon \in H^1(\mathcal{O}_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ and $\frac{\partial}{\partial r}T^\varepsilon u_\varepsilon \rightharpoonup \frac{\partial u}{\partial r}$ weakly in $L^2(\mathcal{O}_U)$. Then,

$$\widetilde{u}_\varepsilon \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} u\, d\tau \quad \text{and} \quad \widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} \frac{\partial u}{\partial r}\, d\tau \quad \text{weakly in } L^2(\mathcal{O}^+).$$

2.3 Boundary Unfolding Operator

In order to obtain the interface conditions, it is necessary to employ the boundary unfolding operator T_0^ε on Γ^ε, which has been inspired by the pioneering work of Daniel Onofrei, who introduced the boundary unfolding operator on a hyperplane in [53]. For every $\varepsilon > 0$, let us denote the unfolded boundary of Γ^ε by Γ_U, defined by

$$\Gamma_U = \{(r_1, \theta, \tau) : \theta \in (0, 2\pi) \text{ and } \tau \in Y(r_1)\}$$

Define the boundary unfolding operator $T_0^\varepsilon : \{u : \Gamma^\varepsilon \to \mathbb{R}\} \to \{T_0^\varepsilon(u) : \Gamma_U \to \mathbb{R}\}$ as

$$T_0^\varepsilon(u)(r_1, \theta, \tau) = u_\varepsilon\left(r_1, \varepsilon\left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right).$$

Note that $T_0^\varepsilon(u) = T^\varepsilon(u)|_{r=r_1}$. Boundary unfolding operator also has similar properties as those of unfolding operator.

2.4 Homogenization of a Semi-Linear Elliptic PDE

In this section, we establish the homogenization of a semi-linear elliptic PDE in $\mathcal{O}_\varepsilon$. We are not writing the measure while doing integration in the article. It is just for getting the expressions in a simple form. If we are taking the functions in polar coordinates, then the integration is with respect to the measure $rdrd\theta$; otherwise, it is with respect to the usual Lebesgue Measure. When we are integrating over the unfolded domain, it is convenient to consider the functions in polar coordinates.

Let $A(r,\theta) = [a_{i,j}(r,\theta)]_{2\times 2}$ be a 2×2 matrix where the entries $a_{ij} : \mathcal{O} \to \mathbb{R}$ are Caratheodory type functions, that is a_{ij} for $i, j = 1, 2$ are measurable in r and continuous in θ. We also assume that $a_{i,j}$ are 2π-periodic with respect to θ and A is uniformly elliptic and bounded in $\mathcal{O}$, that is, there exist constants $\alpha, \beta > 0$ such that

$$\langle A(r,\theta)\lambda, \lambda\rangle \geq \alpha|\lambda|^2 \quad \text{and} \quad |A(r,\theta)\lambda| \leq \beta|\lambda|$$

for all $\lambda \in \mathbb{R}^2$ and $a.e$ in $\mathcal{O}$. Let $k : \mathbb{R} \to \mathbb{R}$ be a C^2 real-valued function such that

$$0 < C_1 \leq k'(t) \leq C_2, k(0) = 0 \text{ and } k'' \text{ is bounded.}$$

Define

$$A^\varepsilon(r, \theta) = [a_{ij}^\varepsilon(r, \theta)]_{2\times 2} = \begin{cases} A\left(r, \frac{\theta}{\varepsilon}\right) & \text{if } (r, \theta) \in \mathcal{O}^+, \\ A(r, \theta) & \text{if } (r, \theta) \in \mathcal{O}^-. \end{cases}$$

Consider the following problem in the domain $\mathcal{O}_\varepsilon$:

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) + k(u_\varepsilon) + u_\varepsilon = f \text{ in } \mathcal{O}_\varepsilon, \\ \qquad\qquad A^\varepsilon \nabla u_\varepsilon \cdot \nu^\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon. \end{cases} \tag{1}$$

Here $f \in L^2(\mathcal{O})$ is a given function, ν^ε is the outward normal vector on $\partial\mathcal{O}_\varepsilon$. The variational form corresponding to (1) is given as: Find $u_\varepsilon \in H^1(\mathcal{O}_\varepsilon)$ such that

$$\int_{\mathcal{O}_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v + k(u_\varepsilon)v + u_\varepsilon v = \int_{\mathcal{O}_\varepsilon} fv \quad \text{for all } v \in H^1(\mathcal{O}_\varepsilon). \tag{2}$$

Since the oscillations are circular, to study the asymptotic behavior, we need to write (2) in polar form as follows:

$$\int_{\mathcal{O}_\varepsilon^+} \left(\bar{A}^\varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial \theta} \end{bmatrix} + k(u_\varepsilon)v + u_\varepsilon v \right) + \int_{\mathcal{O}^-} A \nabla u_\varepsilon \nabla v + u_\varepsilon v = \int_{\mathcal{O}_\varepsilon} fv, \tag{3}$$

for all $v \in H^1(\mathcal{O}_\varepsilon)$, with $\bar{A}^\varepsilon = [\bar{a}_{ij}^\varepsilon]_{2\times 2} = X^t A^\varepsilon X$, where

$$X = \begin{bmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}. \tag{4}$$

Since A^ε is coercive, the matrix $\bar{A}^\varepsilon$ is also coercive. By properties of unfolding operator (see Subsect. 2.2),

$$T^\varepsilon(\bar{A}^\varepsilon) = T^\varepsilon(X^t)T^\varepsilon(A^\varepsilon)T^\varepsilon(X) = T^\varepsilon(X^t)A(r, \tau)T^\varepsilon(X).$$

Then as $\varepsilon \to 0$, it is easy to see the following strong convergence in $L^2(\mathcal{O}_U)$,

$$T^\varepsilon\left(\bar{A}^\varepsilon\right) \to \bar{A} = [\bar{a}_{ij}]_{2\times 2} := X^t A(r, \tau) X.$$

For each $\varepsilon > 0$, we have the existence of unique $u_\varepsilon \in H^1(\mathcal{O}_\varepsilon)$ by the Browder–Minty theorem (see [54]). We want to study the asymptotic behavior of u_ε as $\varepsilon \to 0$. Let us describe the limit problem.

Limit problem: To define the solution of the homogenized variational form, we need appropriate function spaces, which we will define now. For any function ϕ defined on $\mathcal{O}$, we may write $\phi = \phi^+ \chi_{\mathcal{O}^+} + \phi^- \chi_{\mathcal{O}^-} = (\phi^+, \phi^-)$ throughout this article. Define

$$V(\mathcal{O}) = \left\{ \psi \in L^2(\mathcal{O}) : (x \cdot \nabla \psi) \in L^2(\mathcal{O}) \text{ and } \psi \in H^1(\mathcal{O}^-) \right\},$$

with the inner product

$$\langle \phi, \psi \rangle_{V(\mathcal{O})} = \langle \phi, \psi \rangle_{L^2(\mathcal{O}^+)} + \langle (x \cdot \nabla \phi), (x \cdot \nabla \psi) \rangle_{L^2(\mathcal{O}^+)} + \langle \phi, \psi \rangle_{H^1(\mathcal{O}^-)}.$$

Note that since x is strictly away from the origin, $V(\mathcal{O})$ is a Hilbert space. Now we are in a position to define the limit problem:

$$\begin{cases} -\text{div}\left(\dfrac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \right) + |Y(|x|)|(k(u^+) + u^+) = |Y(|x|)| f & \text{in} \quad \mathcal{O}^+, \\ -\text{div}\left(A \nabla u^- \right) + u^- = f & \text{in} \quad \mathcal{O}^-, \\ \dfrac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \cdot \nu = 0 & \text{on} \quad \Gamma_a, \\ A \nabla u^- \cdot \nu = 0 & \text{on} \quad \Gamma_b, \\ u^+ = u^-, \quad \dfrac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \cdot \nu = A \nabla u^- \cdot \nu & \text{on} \quad \Gamma_0, \end{cases} \tag{5}$$

where the limit coefficient a_0 is

$$a_0(r, \theta) = \int_{Y(r)} \left(\frac{\det(A(r, \tau))}{A(r, \tau) \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}} \right) d\tau.$$

The weak form of the limit problem (5) is given by: Find $u = u^+ \chi_{\mathcal{O}^+} + u^- \chi_{\mathcal{O}^-} \in V(\mathcal{O})$ such that

$$\begin{aligned} &\int_{\mathcal{O}^+} \frac{a_0(x)}{|x|^2} (x \cdot \nabla u)(x \cdot \nabla \phi) + |Y(|x|)|(k(u) + u)\phi + \int_{\mathcal{O}^-} A \nabla u \nabla \phi + (k(u) + u)\phi \\ &= \int_{\mathcal{O}^+} |Y(|x|)| f \phi + \int_{\mathcal{O}^-} f \phi, \quad \text{for all } \phi \in V(\mathcal{O}). \end{aligned} \tag{6}$$

Since A is coercive, a_0 is strictly positive, and k is monotone, it follows by Browder–Minty theorem, we have the existence and uniqueness of the solution to the variational form (6) in $V(\mathcal{O})$.

Using the polar transformation $r\frac{\partial}{\partial r}u = (x \cdot \nabla u)$, we can write the polar form of (6) as: Given $f \in L^2(\mathcal{O})$, find $u \in V(\mathcal{O})$ such that

$$\begin{aligned}&\int_{\mathcal{O}^+}\left(a_0\frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r} + Y(r)(k(u)+u)\phi\right) + \int_{\mathcal{O}^-}(A\nabla u\nabla\phi + (k(u)+u)\phi)\\ &= \int_{\mathcal{O}^+} Y(r)f\phi + \int_{\mathcal{O}^-} f\phi, \text{ for all } \phi \in V(\mathcal{O}).\end{aligned} \tag{7}$$

We will now prove the main theorem of this section, which states that the system (5) is the homogenized limit problem of (1). To do this, we will prove the convergence of solutions in their respective polar forms.

Theorem 1 *Let u_ε and u be the unique solutions of (3) and (7) respectively. Then, we have the following convergences weakly in $L^2(\mathcal{O}^+)$*

$$\widetilde{u_\varepsilon} \rightharpoonup |Y(r)|u, \quad \widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup |Y(r)|\frac{\partial u}{\partial r}, \quad \widetilde{\frac{\partial u_\varepsilon}{\partial \theta}} \rightharpoonup \left(-\frac{1}{2\pi}\int_{Y(r)}\frac{\bar{a}_{21}}{\bar{a}_{22}}d\tau\right)\frac{\partial u}{\partial r}$$
$$\text{and} \quad \widetilde{k(u_\varepsilon)} \rightharpoonup |Y(r)|k(u).$$

And in $H^1(\mathcal{O}^-)$, we have

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(\mathcal{O}^-).$$

Proof We are dividing the proof into several steps.
Step 1: (Convergences) Since $\|u_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leq \|f\|_{L^2(\mathcal{O})}$, using the properties of unfolding operator, we have $\{T^\varepsilon(u_\varepsilon)\}$, $\left\{T^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial r}\right)\right\}$ and $\left\{T^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial \theta}\right)\right\}$ are bounded in $L^2(\mathcal{O}_U)$. Also $\{u_\varepsilon\}$ is bounded in $H^1(\mathcal{O}^-)$. Hence from weak compactness, there exist $u^+, P_1, P_2, \zeta \in L^2(\mathcal{O}_U)$ and $u^- \in H^1(\mathcal{O}^-)$ such that

$$\begin{aligned}&T^\varepsilon u_\varepsilon \rightharpoonup u^+, \quad T^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial r}\right) \rightharpoonup P_1, \quad T^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial \theta}\right) \rightharpoonup P_2,\\ &T^\varepsilon(k(u_\varepsilon)) \rightharpoonup \zeta \quad \text{weakly in } L^2(\mathcal{O}_U) \quad \text{and}\\ &u_\varepsilon \longrightarrow u^- \text{ weakly in } H^1(\mathcal{O}^-).\end{aligned} \tag{8}$$

From the properties of unfolding, it is easy to see that

$$P_1 = \frac{\partial u^+}{\partial r}.$$

Using similar properties, we get

$$\frac{\partial}{\partial \tau}T^\varepsilon(u_\varepsilon) \rightharpoonup \frac{\partial u^+}{\partial \tau} \text{ weakly in } L^2(\mathcal{O}_U).$$

But

$$\frac{\partial}{\partial \tau} T^{\varepsilon}(u_{\varepsilon}) = \varepsilon T^{\varepsilon}\left(\frac{\partial u_{\varepsilon}}{\partial \theta}\right) \to 0 \text{ strongly in } L^2(\mathcal{O}_U),$$

which implies u^+ is independent of τ. To identify P_2, choose $\phi \in D(\mathcal{O}^+)$, $\psi \in C^{\infty}([0, 2\pi])$ as arbitrary and define ϕ^{ε} as

$$\phi^{\varepsilon}(r, \theta) = \varepsilon \phi(r, \theta) \psi\left(\left\{\frac{\theta}{\varepsilon}\right\}\right).$$

Then

$$\begin{aligned} &T^{\varepsilon}(\phi^{\varepsilon}) = \varepsilon T^{\varepsilon}(\phi)\psi(\tau), \quad T^{\varepsilon}\left(\frac{\partial \phi^{\varepsilon}}{\partial r}\right) = \varepsilon T^{\varepsilon}\left(\frac{\partial \phi}{\partial r}\right)\psi(\tau) \quad \text{and} \\ &T^{\varepsilon}\left(\frac{\partial \phi^{\varepsilon}}{\partial \theta}\right) = \varepsilon T^{\varepsilon}\left(\frac{\partial \phi}{\partial r}\right) + T^{\varepsilon}(\phi)\, \nabla_y \psi(\tau). \end{aligned} \tag{9}$$

Use ϕ^{ε} as a test function in (3) to get

$$\int_{\mathcal{O}_{\varepsilon}^+} A^{\varepsilon} \nabla u_{\varepsilon} \nabla \phi_{\varepsilon} + k(u_{\varepsilon})\phi_{\varepsilon} + u_{\varepsilon}\phi_{\varepsilon} = \int_{\mathcal{O}_{\varepsilon}^+} f \phi_{\varepsilon}.$$

Apply the unfolding operator and passing to the limit using (8) and (9), we get

$$\int_{\mathcal{O}_U} \bar{A} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} 0 \\ \phi\psi'(\tau) \end{bmatrix} = \int_{\mathcal{O}_U} (\bar{a}_{21} P_1 + \bar{a}_{22} P_2)\, \phi \psi'(\tau) = 0,$$

which implies

$$P_2 = -\frac{\bar{a}_{21}}{\bar{a}_{22}} P_1 = -\frac{\bar{a}_{21}}{\bar{a}_{22}} \frac{\partial u^+}{\partial r}.$$

Step 2: (Interface Condition) Now, we prove the trace $u^+ = u^-$ on Γ_0. By the continuity of the trace operator and using properties of the unfolding operator, we get

$$\begin{aligned} \int_{\Gamma_0} u^+ \phi &= \lim_{\varepsilon \to 0} \int_{\Gamma_0} \left(T^{\varepsilon}(u_{\varepsilon})\right)\big|_{x_n=0} T_0^{\varepsilon}(\phi) = \lim_{\varepsilon \to 0} \int_{\Gamma_0} \left(T_0^{\varepsilon}(u_{\varepsilon}|_{\mathcal{O}^+})\right)\big|_{x_n=0} T_0^{\varepsilon}(\phi) \\ &= \lim_{\varepsilon \to 0} \int_{\Gamma_0} \left(T_0^{\varepsilon}(u_{\varepsilon}|_{\mathcal{O}^-})\right)\big|_{x_n=0} T_0^{\varepsilon}(\phi) = \int_{\Gamma_0} u^- \phi \end{aligned}$$

for any $\phi \in C_c^{\infty}(\Gamma_0)$. Hence, we have $u^+ = u^-$ on Γ_0. Define

$$u = \chi_{\mathcal{O}^+} u^+ + \chi_{\mathcal{O}^-} u^-.$$

Since $\frac{\partial u^+}{\partial r} \in L^2(\mathcal{O}^+)$ and $u^- \in H^1(\mathcal{O}^-)$, the interface condition gives $u \in V(\mathcal{O})$.

Step 3: (Evaluating ζ**)** The calculation of ζ is a crucial aspect of this article that requires delicate analysis. To perform the calculation, we will utilize the well-known Browder–Minty method. Let $\phi \in C^1(\bar{\mathcal{O}})$. Consider the integral

$$\begin{aligned} I_\varepsilon &= \int_{\mathcal{O}_\varepsilon^+} \bar{A}^\varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial \phi}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} - \left(-\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\right)\frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial \phi}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} - \left(-\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\right)\frac{\partial u}{\partial r} \end{bmatrix} \\ &\quad + \int_{\mathcal{O}_\varepsilon^+} (k(u_\varepsilon) - k(\phi)(u_\varepsilon - \phi)) + (u_\varepsilon - \phi)^2 \\ &\quad + \int_{\mathcal{O}^-} A(\nabla u_\varepsilon - \nabla\phi)(\nabla u_\varepsilon - \nabla\phi) + (k(u_\varepsilon) - k(\phi)(u_\varepsilon - \phi)) + (u_\varepsilon - \phi)^2. \end{aligned}$$

Expand and rearrange to get,

$$\begin{aligned} I_\varepsilon &= \int_{\mathcal{O}_\varepsilon} A\nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2 + \int_{\mathcal{O}_\varepsilon^+} -\bar{A}^\varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\frac{\partial u}{\partial r} \end{bmatrix} \\ &\quad + \int_{\mathcal{O}_\varepsilon^+} -\bar{A}^\varepsilon \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} + \int_{\mathcal{O}_\varepsilon^+} \bar{A}^\varepsilon \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\frac{\partial u}{\partial r} \end{bmatrix} \\ &\quad + \int_{\mathcal{O}_\varepsilon} -k(u_\varepsilon)\phi - k(\phi)u_\varepsilon + k(\phi)\phi - 2u_\varepsilon\phi + \phi^2 \\ &\quad + \int_{\mathcal{O}^-} -A\nabla u_\varepsilon\nabla\phi - A\nabla\phi\nabla u_\varepsilon + A\nabla\phi\nabla\phi - k(u_\varepsilon)\phi - k(\phi)u_\varepsilon + k(\phi)\phi - 2u_\varepsilon\phi + \phi^2. \end{aligned}$$

Now we have to pass the limit as $\varepsilon \to 0$. Using (8) pass to the limit as $\varepsilon \to 0$ in the variational form (3) to get

$$\begin{aligned} \int_{\mathcal{O}_U} f\phi + \int_{\mathcal{O}^-} f\phi &= \int_{\mathcal{O}_U} \bar{A} \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r} \end{bmatrix} \nabla\phi + \zeta\phi + u\phi + \int_{\mathcal{O}^-} A\nabla u\nabla\phi + k(u)\phi + u\phi \\ &= \int_{\mathcal{O}_U} \left(\frac{\det\bar{A}}{\bar{a}_{22}}\right)\frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r} + \zeta\phi + u\phi + \int_{\mathcal{O}^-} A\nabla u\nabla\phi + k(u)\phi + u\phi. \end{aligned}$$

By density of $C^1(\bar{\mathcal{O}})$ in $V(\mathcal{O})$, the above equality holds for all $\phi \in V(\mathcal{O})$. Put $\phi = u$, we have

$$\begin{aligned} \int_{\mathcal{O}_U} fu + \int_{\mathcal{O}^-} fu &= \int_{\mathcal{O}_U} \left(\frac{\det\bar{A}}{\bar{a}_{22}}\right)\frac{\partial u}{\partial r}\frac{\partial u}{\partial r} + \zeta u + u^2 + \int_{\mathcal{O}^-} A\nabla u\nabla u + k(u)u + u^2 \\ &= \int_{\mathcal{O}_U} \bar{A} \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r} \end{bmatrix} + \zeta u + u^2 \\ &\quad + \int_{\mathcal{O}^-} A\nabla u\nabla u + k(u)u + u^2. \end{aligned}$$

On the other hand using the energy equality, we get

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2 = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\varepsilon} f u_\varepsilon = \int_{\mathcal{O}_U} f u + \int_{\mathcal{O}^-} f u$$
$$= \int_{\mathcal{O}_U} \bar{A}\begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix} + \zeta u + u^2 + \int_{\mathcal{O}^-} A\nabla u \nabla u + k(u)u + u^2. \tag{10}$$

Now using (8) and (10), we get (re-ordered for convenience)

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon &= \int_{\mathcal{O}_U} \bar{A}\begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix} - \int_{\mathcal{O}_U} \bar{A}\begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix}\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix} \\
&\quad - \int_{\mathcal{O}_U} \bar{A}\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix} + \int_{\mathcal{O}_U} \bar{A}\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix}\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix} \\
&\quad + \int_{\mathcal{O}_U} \zeta u - \zeta\phi - k(\phi)u + k(\phi)\phi + u^2 - 2u\phi + \phi^2 \\
&\quad + \int_{\mathcal{O}^-} A\nabla u\nabla u - A\nabla u\nabla\phi - A\nabla\phi\nabla u + A\nabla\phi\nabla\phi \\
&\quad + \int_{\mathcal{O}^-} k(u)u - k(u)\phi - k(\phi)u + k(\phi)\phi + u^2 - 2u\phi + \phi^2.\end{aligned}$$

By performing proper factorization, we arrive at

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon &= \int_{\mathcal{O}_U} A\begin{bmatrix} \frac{\partial u}{\partial r} - \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r} + \frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial r} - \frac{\partial \phi}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r} + \frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r}\end{bmatrix} \\
&\quad + \int_{\mathcal{O}_U} (\zeta - k(u))(u-\phi) + (u-\phi)^2 \\
&\quad + \int_{\mathcal{O}^-} A(\nabla u - \nabla\phi)(\nabla u - \nabla\phi) + (k(u) - k(\phi))(u-\phi) + (u-\phi)^2 \\
&= \int_{\mathcal{O}_U} \bar{a}_{11}\left(\frac{\partial u}{\partial r} - \frac{\partial \phi}{\partial r}\right)\left(\frac{\partial u}{\partial r} - \frac{\partial \phi}{\partial r}\right) + (\zeta - k(\phi))(u-\phi) + (u-\phi)^2 \\
&\quad + \int_{\mathcal{O}^-} A(\nabla u - \nabla\phi)(\nabla u - \nabla\phi) + (k(u) - k(\phi))(u-\phi) + (u-\phi)^2.\end{aligned}$$

From the monotonicity of k, we have $I^\varepsilon \geq 0$ for all ε, which implies

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon &= \int_{\mathcal{O}_U} \bar{a}_{11}\left(\frac{\partial u}{\partial r} - \frac{\partial \phi}{\partial r}\right)\left(\frac{\partial u}{\partial r} - \frac{\partial \phi}{\partial r}\right) + (\zeta - k(\phi))(u-\phi) + (u-\phi)^2 \\
&\quad + \int_{\mathcal{O}^-} A(\nabla u - \nabla\phi)(\nabla u - \nabla\phi) + (k(u) - k(\phi))(u-\phi) + (u-\phi)^2 \geq 0.\end{aligned}$$

The above inequality holds true for all $\phi \in V(\mathcal{O})$. At this stage, choose $\phi = u + \lambda\psi$, $\psi \in V(\mathcal{O})$, $\lambda > 0$ to get

$$\int_{\mathcal{O}_U} \lambda \bar{a}_{11} \frac{\partial \psi}{\partial r}\frac{\partial \psi}{\partial r} + (\zeta - k(\phi - \lambda\psi))\psi + \lambda\psi^2 + \int_{\mathcal{O}^-} \lambda A \nabla\psi\nabla\psi + (k(u) - k(u - \lambda\psi))\psi + \lambda\psi^2 \geq 0.$$

As $\lambda \to 0$,

$$\int_{\mathcal{O}_U} (\zeta - k(u))\,\psi \geq 0 \text{ for all } \psi \in V(\mathcal{O}).$$

Hence,

$$\int_{Y(r)} \zeta\, dy = |Y(r)| k(u). \tag{11}$$

Thus, we have evaluated all the unknowns in (8). Hence using properties of the unfolding operator, we can deduce the following convergences weakly in $L^2(\mathcal{O}^+)$

$$\widetilde{u_\varepsilon} \rightharpoonup |Y(r)|u, \qquad \widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup |Y(r)|\frac{\partial u}{\partial r}, \qquad \widetilde{\frac{\partial u_\varepsilon}{\partial \theta}} \rightharpoonup \left(-\frac{1}{2\pi}\int_{Y(r)} \frac{\bar{a}_{21}}{\bar{a}_{22}} d\tau\right)\frac{\partial u}{\partial r}$$
$$\text{and} \quad \widetilde{k(u_\varepsilon)} \rightharpoonup |Y(r)|k(u).$$

Hence we got the required convergence. Now we need to prove that u is actually the solution of the limit problem (7).

Step 4: (Limit Problem) Use $\psi \in C^\infty(\bar{\mathcal{O}})$ as a test function in (3). Apply unfolding operator and passing to the limit in (3) using (8), we obtain

$$\int_{\mathcal{O}_U} \bar{A} \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}}\frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi}{\partial r} \\ \frac{\partial \psi}{\partial \theta} \end{bmatrix} + \zeta\psi + u\psi + \int_{\mathcal{O}^-} A\nabla u \nabla\psi + k(u)\psi + u\psi = \int_{\mathcal{O}_U} f\psi + \int_{\mathcal{O}^-} f\psi.$$

Simplify to get,

$$\int_{\mathcal{O}_U} \left(\frac{\det \bar{A}}{\bar{a}_{22}}\right)\frac{\partial u}{\partial r}\frac{\partial \psi}{\partial r} + \zeta\psi + u\psi + \int_{\mathcal{O}^-} A\nabla u\nabla\psi + k(u)\psi + u\psi = \int_{\mathcal{O}_U} f\psi + \int_{\mathcal{O}^-} f\psi.$$

Average out using (11) and properties of the unfolding operator to get

$$\int_{\mathcal{O}^+} a_0 \frac{\partial u}{\partial r}\frac{\partial \psi}{\partial r} + |Y(|x|)|k(u)\psi + u\psi + \int_{\mathcal{O}^-} A\nabla u \nabla \psi + k(u)\psi + u\psi$$
$$= \int_{\mathcal{O}^+} |Y(r)| f\psi + \int_{\mathcal{O}^-} f\psi,$$

where

$$a_0 = \int_{Y(r)} \left(\frac{\det \bar{A}}{\bar{a}_{22}}\right) d\tau = \int_{Y(r)} \left(\frac{\det(A(r,\tau))}{A(r,\tau)\begin{bmatrix} -\sin(\theta) \\ \cos(\theta)\end{bmatrix}\begin{bmatrix} -\sin(\theta) \\ \cos(\theta)\end{bmatrix}} \right) d\tau.$$

By density of $C^\infty(\mathcal{O})$, in $V(\mathcal{O})$, we get that u satisfies the limit problem (7). Hence the proof of Theorem 1 is done. □

As we proceed, we will prove the following corrector results (strong convergences), which are crucial in proving homogenization of optimal control problems in next section.

Theorem 2 *Let u_ε and u be the unique solutions of (3) and (7) respectively. Then as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u\|_{L^2(\mathcal{O}_\varepsilon^+)} + \left\| \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial u}{\partial r} \right\|_{L^2(\mathcal{O}_\varepsilon^+)} + \left\| \frac{\partial u_\varepsilon}{\partial \theta} + \frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\frac{\partial u}{\partial r} \right\|_{L^2(\mathcal{O}_\varepsilon^+)}$$
$$+ \|u_\varepsilon - u\|_{H^1(\mathcal{O}^-)} \to 0.$$

Proof Consider

$$J_\varepsilon = \int_{\mathcal{O}_\varepsilon^+} \bar{A}_\varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial u}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} - \left(-\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\frac{\partial u}{\partial r}\right) \end{bmatrix}\begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial u}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} - \left(-\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon}\frac{\partial u}{\partial r}\right) \end{bmatrix}$$
$$+ \int_{\mathcal{O}_\varepsilon^+} (k(u_\varepsilon) - k(u))(u_\varepsilon - u) + (u_\varepsilon - u)^2$$
$$+ \int_{\mathcal{O}^-} A(\nabla u_\varepsilon - \nabla u)(\nabla u_\varepsilon - \nabla u) + (k(u_\varepsilon) - k(u))(u_\varepsilon - u) + (u_\varepsilon - u)^2.$$

Expand and rearrange to get

$$J_\varepsilon = J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3 + J_\varepsilon^4,$$

where

$$J_\varepsilon^1 = \int_{\mathcal{O}_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2,$$

$$
\begin{aligned}
J_\varepsilon^2 &= \int_{\mathcal{O}_\varepsilon^+} -A \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon} \frac{\partial u}{\partial r} \end{bmatrix} - A \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon} \frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} + A \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon} \frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial r} \\ -\frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon} \frac{\partial u}{\partial r} \end{bmatrix}, \\
J_\varepsilon^3 &= \int_{\mathcal{O}_\varepsilon^+} -k(u_\varepsilon)u - k(u)u_\varepsilon + k(u)u - 2u_\varepsilon u + u^2, \\
J_\varepsilon^4 &= \int_{\mathcal{O}^-} -A\nabla u_\varepsilon \nabla u - A\nabla u \nabla u_\varepsilon + A\nabla u \nabla u - k(u_\varepsilon)u - k(u)u_\varepsilon + k(u)u - 2u_\varepsilon u + u^2.
\end{aligned}
$$

On applying the unfolding operator and passing to the limit as $\varepsilon \to 0$, we get

$$
\begin{aligned}
\lim_{\varepsilon \to 0} J_\varepsilon^2 &= \int_{\mathcal{O}_U} \left(\bar{a}_{12} \frac{\bar{a}_{21}}{\bar{a}_{22}} - \bar{a}_{11} \right) \frac{\partial u}{\partial r} \frac{\partial u}{\partial r} = \int_{\mathcal{O}^+} -a_0 \frac{\partial u}{\partial r} \frac{\partial u}{\partial r}, \\
\lim_{\varepsilon \to 0} J_\varepsilon^3 &= \int_{\mathcal{O}_U} -\zeta u - u^2 = \int_{\mathcal{O}^+} -|Y(r)| \left(k(u)u + u^2 \right), \\
\lim_{\varepsilon \to 0} J_\varepsilon^4 &= \int_{\mathcal{O}^-} -A\nabla u \nabla u - k(u)u - u^2, \\
\lim_{\varepsilon \to 0} J_\varepsilon^1 &= \lim_{\varepsilon \to 0} \int_{\mathcal{O}_\varepsilon} A\nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2 \\
&= \lim_{\varepsilon \to 0} \int_{\mathcal{O}_\varepsilon} f u_\varepsilon = \int_{\mathcal{O}_U} f u + \int_{\mathcal{O}^-} f u \\
&= \int_{\mathcal{O}^+} a_0 \frac{\partial u}{\partial r} \frac{\partial u}{\partial r} + |Y(r)| \left(k(u)u + u^2 \right) + \int_{\mathcal{O}^-} A\nabla u \nabla u + k(u)u + u^2 \\
&= -\left(\lim_{\varepsilon \to 0} J_2^\varepsilon + \lim_{\varepsilon \to 0} J_3^\varepsilon + \lim_{\varepsilon \to 0} J_4^\varepsilon \right).
\end{aligned}
$$

This implies that $\lim_{\varepsilon \to 0} J_\varepsilon = 0$. Then coercivity of A and monotonicity of k completes the proof of Theorem 2. □

2.5 Homogenization of Optimal Control Problem

Here we are going to study an optimal control problem in $\mathcal{O}_\varepsilon$ governed by a semi-linear elliptic PDE described in the previous section. Let $A(r,\theta) = [a_{i,j}(r,\theta)]_{2\times 2}$ and $B(r,\theta) = [b_{i,j}(r,\theta)]_{2\times 2}$ be 2×2 symmetric matrices that are uniformly elliptic, bounded and 2π-periodic with respect to the variable θ . Also, the entries $a_{ij}, b_{ij} : \mathcal{O} \to \mathbb{R}$ are Caratheodory type functions that is measurable in r and continuous in θ. Define

$$
A^\varepsilon(r,\theta) = [a_{ij}^\varepsilon(r,\theta)]_{2\times 2} = \begin{cases} A\left(r, \frac{\theta}{\varepsilon}\right) & \text{if } (r,\theta) \in \mathcal{O}^+, \\ A(r,\theta) & \text{if } (r,\theta) \in \mathcal{O}^-. \end{cases}
$$

$$
B^\varepsilon(r,\theta) = [b_{ij}^\varepsilon(r,\theta)]_{2\times 2} = \begin{cases} B\left(r, \frac{\theta}{\varepsilon}\right) & \text{if } (r,\theta) \in \mathcal{O}^+, \\ B(r,\theta) & \text{if } (r,\theta) \in \mathcal{O}^-. \end{cases}
$$

As we defined in the previous section, define $\bar{A}_\varepsilon = X^t A^\varepsilon X$ and $\bar{B}_\varepsilon = X^t B^\varepsilon X$, where X is given by (4). As we discussed in the previous section, as $\varepsilon \to 0$, it is easy to see the following strong convergence in $L^2(\mathcal{O}_U)$,

$$T^\varepsilon\left(\bar{A}^\varepsilon\right) \to \bar{A} = [\bar{a}_{ij}]_{2\times 2} := X^t A(r,\tau)X,$$
$$T^\varepsilon\left(\bar{B}^\varepsilon\right) \to \bar{B} = [\bar{b}_{ij}]_{2\times 2} := X^t B(r,\tau)X.$$

Let $\omega \subset\subset \mathcal{O}^-$ be an open set and admissible control set is $L^2(\omega)$. Consider the following minimization problem in $\mathcal{O}_\varepsilon$

$$\text{Minimize: } J_\varepsilon(u,\theta) = \frac{1}{2}\int_{\mathcal{O}_\varepsilon} B^\varepsilon \nabla u \nabla u + \frac{\beta}{2}\int_{\mathcal{O}_\varepsilon} \chi_\omega |\theta|^2, \tag{12}$$

where (u,θ) satisfies the following system,

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u) + k(u) + u = f + \chi_\omega \theta \text{ in } \mathcal{O}_\varepsilon, \\ \qquad\qquad A^\varepsilon \nabla u \cdot \nu_\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon, \end{cases}$$

with $f \in L^2(\mathcal{O})$. One of the aspects is the consideration of the cost functional by a different matrix B. Even for such a problem in fixed domain, the homogenization analysis is delicate. Let us recall the following well-known result on semi-linear optimal control problems (see [15, 55]).

Theorem 3 *Let* $(u_\varepsilon, \theta_\varepsilon)$ *be the unique solution of* (12). *Then the optimality system is given by*

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) + k(u_\varepsilon) + u_\varepsilon = f + \chi_\omega \theta_\varepsilon \text{ in } \mathcal{O}_\varepsilon, \\ -\operatorname{div}(A^\varepsilon \nabla v_\varepsilon) + k'(u_\varepsilon)v_\varepsilon + v_\varepsilon = -\operatorname{div}(B^\varepsilon \nabla u_\varepsilon) \text{ in } \mathcal{O}_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon = 0, \quad A^\varepsilon \nabla v_\varepsilon \cdot \nu_\varepsilon = B^\varepsilon \nabla u_\varepsilon \text{ on } \partial\mathcal{O}_\varepsilon, \\ \theta_\varepsilon = -\chi_\omega \dfrac{1}{\beta} v_\varepsilon. \end{cases} \tag{13}$$

To be precise, v_ε is the adjoint state. The variational formulation for the optimality system (13) is as follows: Given $f \in L^2(\mathcal{O})$, find $(u_\varepsilon, v_\varepsilon) \in H^1(\mathcal{O}_\varepsilon) \times H^1(\mathcal{O}_\varepsilon)$ such that

$$\begin{cases} \displaystyle\int_{\mathcal{O}_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla \phi + (k(u_\varepsilon) + u_\varepsilon)\phi = \int_{\mathcal{O}_\varepsilon} (f + \chi_\omega \theta_\varepsilon)\phi, \\ \displaystyle\int_{\mathcal{O}_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla \psi + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)\psi = \int_{\mathcal{O}_\varepsilon} B^\varepsilon \nabla u_\varepsilon \nabla \psi, \end{cases} \tag{14}$$

for all $(\phi, \psi) \in H^1(\mathcal{O}_\varepsilon) \times H^1(\mathcal{O}_\varepsilon)$ with $\theta_\varepsilon = -\frac{1}{\beta}\chi_\omega v_\varepsilon$.

We want to study the asymptotic behavior of $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0$. Since the oscillations are in a circular fashion, we rewrite (14) in polar form as:

$$\begin{cases} \displaystyle\int_{\mathcal{O}_\varepsilon^+} \left(\bar{A}^\varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial \theta} \end{bmatrix} + (k(u_\varepsilon) + u_\varepsilon)\phi \right) + \int_{\mathcal{O}^-} A\nabla u_\varepsilon \nabla \phi + (k(u_\varepsilon) + u_\varepsilon)\phi \\ \quad = \displaystyle\int_{\mathcal{O}_\varepsilon} (f + \chi_\omega \theta_\varepsilon)\phi, \\ \displaystyle\int_{\mathcal{O}_\varepsilon^+} \left(\bar{A}^\varepsilon \begin{bmatrix} \frac{\partial v_\varepsilon}{\partial r} \\ \frac{\partial v_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi}{\partial r} \\ \frac{\partial \psi}{\partial \theta} \end{bmatrix} + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)\psi \right) + \int_{\mathcal{O}^-} A\nabla v_\varepsilon \nabla \psi + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)\psi \\ \quad = \displaystyle\int_{\mathcal{O}_\varepsilon^+} \left(\bar{B}^\varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi}{\partial r} \\ \frac{\partial \psi}{\partial \theta} \end{bmatrix} \right) + \int_{\mathcal{O}^-} B\nabla u_\varepsilon \nabla \psi \end{cases} \tag{15}$$

for all $(\phi, \psi) \in H^1(\mathcal{O}_\varepsilon) \times H^1(\mathcal{O}_\varepsilon)$ with $\theta_\varepsilon = -\frac{1}{\beta}\chi_\omega v_\varepsilon$.

We will now describe the limit optimal control problem, which, as we will show in Theorem 5, is the homogenized problem. For the limit problem, the control set is also $L^2(\omega)$. Consider the following minimization problem: Minimize

$$J(u, \theta) = \frac{1}{2} \int_{\mathcal{O}^+} \frac{b_0}{|x|^2} (x \cdot \nabla u^+) \cdot (x \cdot \nabla u^+) + \frac{1}{2} \int_{\mathcal{O}^-} B\nabla u^- \nabla u^- + \frac{\beta}{2} \int_\omega |\theta|^2, \tag{16}$$

where (u, θ) satisfies the following system:

$$\begin{cases} -\operatorname{div}\left(\frac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \right) + |Y(|x|)|(k(u^+) + u^+) = |Y(|x|)| f & \text{in } \mathcal{O}^+, \\ -\operatorname{div}(A\nabla u^-) + k(u^-) + u^- = f + \theta & \text{in } \mathcal{O}^-, \\ \frac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \cdot \nu = 0 & \text{on } \Gamma_a, \\ A\nabla u^- \cdot \nu = 0 & \text{on } \Gamma_b, \\ u^+ = u^-, \quad \frac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \cdot \nu = A\nabla u^- \cdot \nu & \text{on } \Gamma_0. \end{cases}$$

If $(u, \theta) \in V(\mathcal{O}) \times L^2(\omega)$ is the unique optimal solution of the limit minimization problem, it will satisfy the following optimality system

$$
\begin{cases}
-\text{div}\left(\frac{a_0(x)}{|x|^2}\left(x\cdot\nabla u^+\right)x\right)+|Y(|x|)|(k(u^+)+u^+)=|Y(|x|)|f & \text{in } \mathcal{O}^+,\\
-\text{div}\left(\frac{a_0(x)}{|x|^2}\left(x\cdot\nabla v^+\right)x\right)+|Y(x)|(k'(u^+)v^++v^+) & \\
\qquad =-\text{div}\left(\frac{b_0(x)}{|x|^2}\left(x\cdot\nabla u^+\right)x\right) & \text{in } \mathcal{O}^+,\\
-\text{div}(A\nabla u^-)+k(u^-)+u^-=f+\theta & \text{in } \mathcal{O}^-,\\
-\text{div}(A\nabla u^-)+k'(u^-)v^-+v^-=-\text{div}(B\nabla u^-) & \text{in } \mathcal{O}^-,\\
\theta=-\frac{1}{\beta}\chi_\omega v^- & \text{in } \mathcal{O}^-
\end{cases}
$$

together with the boundary conditions

$$
\begin{cases}
\frac{a_0(x)}{|x|^2}\left(x\cdot\nabla u^+\right)x\cdot\nu=0 & \text{on } \Gamma_a,\\
\frac{a_0(x)}{|x|^2}\left(x\cdot\nabla v^+\right)x\cdot\nu=\frac{b_0(x)}{|x|^2}\left(x\cdot\nabla u^+\right)x\cdot\nu & \text{on } \Gamma_a,\\
A\nabla u^-\cdot\nu=0,\quad A\nabla v^-\cdot\nu=B\nabla u^-\cdot\nu & \text{on } \Gamma_b
\end{cases}
$$

and the interface conditions on Γ_0

$$
\begin{cases}
u^+=u^-,\quad v^+=v^-,\quad \frac{a_0(x)}{|x|^2}\left(x\cdot\nabla u^+\right)x\cdot\nu=A\nabla u^-\cdot\nu\\
\frac{a_0(x)}{|x|^2}\left(x\cdot\nabla v^+\right)x\cdot\nu-\frac{b_0(x)}{|x|^2}\left(x\cdot\nabla u^+\right)x\cdot\nu=(A\nabla v^--B\nabla u^-)\cdot\nu.
\end{cases}
$$

Here the coefficients a_0 and b_0 are given by

$$
a_0=\int_{Y(|x|)}\left(\frac{\det\bar{A}}{\bar{a}_{22}}\right)d\tau \quad\text{and}\quad b_0=\int_{Y(|x|)}\left(\bar{B}\begin{bmatrix}1\\-\frac{\bar{a}_{21}}{\bar{a}_{22}}\end{bmatrix}\begin{bmatrix}1\\-\frac{\bar{a}_{21}}{\bar{a}_{22}}\end{bmatrix}\right)d\tau.
$$

Corresponding weak formulation is: Given $f\in L^2(\mathcal{O})$ find $(u,v)\in V(\mathcal{O})\times V(\mathcal{O})$ such that

$$
\begin{cases}
\int_{\mathcal{O}^+} \frac{a_0(x)}{|x|^2}(x \cdot \nabla u)(x \cdot \nabla \psi) + |Y(|x|)|(k(u)+u)\psi \\
\quad + \int_{\mathcal{O}^-} A\nabla u \nabla \psi + (k(u)+u)\psi = \int_{\mathcal{O}^+} |Y(|x|) f\psi + \int_{\mathcal{O}^-} (f + \chi_\omega \theta)\psi, \\
\int_{\mathcal{O}^+} \frac{a_0(x)}{|x|^2}(x \cdot \nabla u)(x \cdot \nabla \phi) + |Y(|x|)|\left(k'(u)v + v\right)\phi \\
\quad + \int_{\mathcal{O}^-} A\nabla v \nabla \phi + (k'(u)v + v)\phi \\
= \int_{\mathcal{O}^+} \frac{b_0(x)}{|x|^2}(x \cdot \nabla u)(x \cdot \nabla \phi) + \int_{\mathcal{O}^-} B\nabla u \nabla \phi,
\end{cases} \tag{17}
$$

for all $(\psi, \phi) \in V(\mathcal{O}) \times V(\mathcal{O})$ with $\theta = -\frac{1}{\beta}\chi_\omega v$.

Note that a_0 is not influenced by the cost functional, whereas the coefficient b_0 in cost functional is not only depends on the cost of the inhomogenized functional, it also influenced by the dynamics A.

Using the polar transformation $r\frac{\partial}{\partial r}u = (x \cdot \nabla u)$, we can write the polar form of (17) as: Given $f \in L^2(\mathcal{O})$ find $(u, v) \in V(\mathcal{O}) \times V(\mathcal{O})$ such that

$$
\begin{cases}
\int_{\mathcal{O}^+} a_0 \frac{\partial u}{\partial r}\frac{\partial \psi}{\partial r} + |Y(r)|(k(u)+u)\psi + \int_{\mathcal{O}^-} A\nabla u \nabla \psi + (k(u)+u)\psi \\
\quad = \int_{\mathcal{O}^+} |Y(r) f\psi + \int_{\mathcal{O}^-} (f + \chi_\omega \theta)\psi, \\
\int_{\mathcal{O}^+} a_0 \frac{\partial v}{\partial r}\frac{\partial \phi}{\partial r} + |Y(r)|\left(k'(u)v + v\right)\phi + \int_{\mathcal{O}^-} A\nabla v \nabla \phi + (k'(u)v + v)\phi \\
\quad = \int_{\mathcal{O}^+} b_0 \frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r} + \int_{\mathcal{O}^-} B\nabla u \nabla \phi,
\end{cases} \tag{18}
$$

for all $(\psi, \phi) \in V(\mathcal{O}) \times V(\mathcal{O})$ with $\theta = -\frac{1}{\beta}\chi_\omega v$.

Also, the limit minimization problem (16) transform into the following: Minimize

$$
J(u, \theta) = \frac{1}{2}\int_{\mathcal{O}^+} b_0 \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{2}\int_{\mathcal{O}^-} B\nabla u^- \nabla u^- + \frac{\beta}{2}\int_\omega |\theta|^2,
$$

where (u, θ) satisfies the following variational form,

$$
\int_{\mathcal{O}^+} a_0 \frac{\partial u}{\partial r}\frac{\partial \psi}{\partial r} + |Y(r)|(k(u)+u)\psi + \int_{\mathcal{O}^-} A\nabla u \nabla \psi + (k(u)+u)\psi
$$
$$
= \int_{\mathcal{O}^+} |Y(r) f\psi + \int_{\mathcal{O}^-} (f + \chi_\omega \theta)\psi.
$$

The definition of a_0 and b_0 implies the coerciveness of a_0 and b_0. We already have monotonicity of k, then by semi-linear optimal control theory (see [6, 15, 55]), we have the existence and uniqueness of the optimal solution $(\bar{u}, \bar{\theta}) \in V(\mathcal{O}) \times L^2(\omega)$ and (18) is optimality system.

Theorem 4 *Let* $(u_\varepsilon, v_\varepsilon)$ *and* (u, v) *be solutions of* (15) *and* (18) *respectively. Then as* $\varepsilon \to 0$, *we have*

$$\|u_\varepsilon - u\|_{L^2(\mathcal{O}_\varepsilon^+)} + \left\| \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial u}{\partial r} \right\|_{L^2(\mathcal{O}_\varepsilon^+)} + \left\| \frac{\partial u_\varepsilon}{\partial \theta} + \frac{\bar{a}_{21}^\varepsilon}{\bar{a}_{22}^\varepsilon} \frac{\partial u}{\partial r} \right\|_{L^2(\mathcal{O}_\varepsilon^+)} + \|u_\varepsilon - u\|_{H^1(\mathcal{O}^-)} \longrightarrow 0.$$

Proof The proof will be the same as we did in Theorem 2. The only extra term is $\chi_\omega \theta_\varepsilon$. Since ω is compactly contained in Ω^-, and $\|\theta^\varepsilon\|_{H^1(\omega)} \le C$. Hence, it won't make any issues in any step of the proof of Theorems 1 and 2. □

Theorem 5 *Let* $(u_\varepsilon, v_\varepsilon)$ *and* (u, v) *be solutions of* (15) *and* (18) *respectively. Then as* $\varepsilon \to 0$, *we have the following convergences weakly in* $L^2(\mathcal{O}^+)$

$$\widetilde{v_\varepsilon} \rightharpoonup |Y(r)| v, \quad \widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup |Y(r)| \frac{\partial u}{\partial r} \quad \text{and}$$
$$\widetilde{\frac{\partial v_\varepsilon}{\partial \theta}} \rightharpoonup \left(\frac{1}{2\pi} \int_{Y(r)} \frac{1}{\bar{a}_{22}} \left(\bar{b}_{21} - \bar{b}_{22} \frac{\bar{a}_{21}}{\bar{a}_{22}} \right) \right) \frac{\partial u}{\partial r} - \left(\frac{1}{2\pi} \int_{Y(r)} \frac{\bar{a}_{21}}{\bar{a}_{22}} \right) \frac{\partial v}{\partial r}.$$

And in $H^1(\mathcal{O}^-)$, *we have*

$$v_\varepsilon \rightharpoonup v \quad \text{weakly in } H^1(\mathcal{O}^-).$$

Proof We are dividing the proof into several steps.
Step 1: (Convergences) Since $\|v_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)}$ is bounded, using the properties of unfolding operator defined in Sect. 3.2, we have $\{T^\varepsilon(v_\varepsilon)\}$, $\left\{ T^\varepsilon \left(\frac{\partial v_\varepsilon}{\partial r} \right) \right\}$ and $\left\{ T^\varepsilon \left(\frac{\partial v_\varepsilon}{\partial \theta} \right) \right\}$ are bounded in $L^2(\Omega_U)$. Also $\{v_\varepsilon\}$ is bounded in $H^1(\mathcal{O}^-)$. Hence from weak compactness, there exist v^+, Q_1, $Q_2 \in L^2(\mathcal{O}_U)$ and $v^- \in H^1(\mathcal{O}^-)$ such that

$$T^\varepsilon v_\varepsilon \rightharpoonup v^+, \quad T^\varepsilon \left(\frac{\partial v_\varepsilon}{\partial r} \right) \rightharpoonup Q_1, \quad T^\varepsilon \left(\frac{\partial v_\varepsilon}{\partial \theta} \right) \rightharpoonup Q_2 \quad \text{weakly in } L^2(\mathcal{O}_U)$$
$$\text{and} \quad v_\varepsilon \longrightarrow v^- \text{ weakly in } H^1(\mathcal{O}^-). \tag{19}$$

From the properties of unfolding, it is easy to see that

$$Q_1 = \frac{\partial v^+}{\partial r}.$$

Now to identify Q_2, choose ϕ^ε defined in (9) as test function in the variational from (17) to get

$$\int_{\mathcal{O}_\varepsilon^+} A^\varepsilon \nabla v_\varepsilon \nabla \phi^\varepsilon + k'(u_\varepsilon) v_\varepsilon \phi^\varepsilon + v_\varepsilon \phi^\varepsilon = \int_{\mathcal{O}_\varepsilon^+} B \nabla u_\varepsilon \nabla \phi^\varepsilon.$$

Apply unfolding and pass to the limit as $\varepsilon \to 0$ using (8) and (19) to get

$$\int_{\mathcal{O}_U} \bar{A} \begin{bmatrix} \frac{\partial v^+}{\partial r} \\ Q_2 \end{bmatrix} \begin{bmatrix} 0 \\ \phi\psi'(\tau) \end{bmatrix} = \int_{\mathcal{O}_U} \bar{B} \begin{bmatrix} \frac{\partial u^+}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \frac{\partial u^+}{\partial r} \end{bmatrix} \begin{bmatrix} 0 \\ \phi\psi'(\tau) \end{bmatrix},$$

which implies

$$Q_2 = \frac{1}{\bar{a}_{22}} \left(\left(\bar{b}_{21} - \bar{b}_{22} \frac{\bar{a}_{21}}{\bar{a}_{22}} \right) \frac{\partial u}{\partial r} - \bar{a}_{21} \frac{\partial v^+}{\partial r} \right). \tag{20}$$

Now from the same arguments as in Step 2 in the proof of Theorem 1, we can prove the interface condition $v^+ = v^-$ on Γ_0. Define $v = \chi_{\mathcal{O}^+} v^+ + \chi_{\mathcal{O}^-} v^-$. Since $\frac{\partial v}{\partial r}^+ \in L^2(\mathcal{O}^+)$ and $v^- \in H^1(\mathcal{O}^-)$, the interface condition gives $v \in V(\mathcal{O})$. Hence using the averaging property of the unfolding operator, we can deduce the following convergence:

$$\begin{aligned}
&\widetilde{v_\varepsilon} \rightharpoonup |Y(r)| v, \quad \widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup |Y(r)| \frac{\partial u}{\partial r}, \\
&\widetilde{\frac{\partial v_\varepsilon}{\partial \theta}} \rightharpoonup \left(\frac{1}{2\pi} \int_{Y(r)} \frac{1}{\bar{a}_{22}} \left(\bar{b}_{21} - \bar{b}_{22} \frac{\bar{a}_{21}}{\bar{a}_{22}} \right) \right) \frac{\partial u}{\partial r} - \left(\frac{1}{2\pi} \int_{Y(r)} \frac{a_{21}}{\bar{a}_{22}} \right) \frac{\partial v}{\partial r} \\
&\text{weakly in } L^2(\Omega^+) \quad \text{and} \quad v_\varepsilon \rightharpoonup v \quad \text{weakly in } H^1(\mathcal{O}^-).
\end{aligned}$$

Step 2: (Limit problem) Now the remaining part is to prove that v solves the limit problem. Take $\psi \in C^\infty(\bar{\mathcal{O}})$ as a test function in the variational form (18), apply unfolding and pass to the limit as $\varepsilon \to 0$ to get

$$\begin{aligned}
&\int_{\mathcal{O}_U} \bar{A} \begin{bmatrix} \frac{\partial v}{\partial r} \\ Q_2 \end{bmatrix} \nabla\psi + k'(u) v\psi + v\psi + \int_{\mathcal{O}^-} A\nabla u \nabla\psi + k'(u) v\psi + v\psi \\
&= \int_{\mathcal{O}_U} \bar{B} \begin{bmatrix} \frac{\partial u^+}{\partial r} \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \frac{\partial u^+}{\partial r} \end{bmatrix} \nabla\psi + \int_{\mathcal{O}^-} \bar{B} \nabla u \nabla\psi.
\end{aligned}$$

Simplify using (20) to get

$$\begin{aligned}
&\int_{\mathcal{O}_U} \left(\frac{\det \bar{A}}{\bar{a}_{22}} \right) \frac{\partial v}{\partial r} \frac{\partial \psi}{\partial r} + k'(u) v\psi + v\psi + \int_{\mathcal{O}^-} A\nabla u \nabla\psi + k'(u) v\psi + v\psi \\
&= \int_{\mathcal{O}_U} \left(\bar{b}_{11} - \frac{\bar{b}_{12}\bar{a}_{21}}{\bar{a}_{22}} - \frac{\bar{a}_{12} b_{21}}{\bar{a}_{22}} + \frac{\bar{a}_{12} b_{22} \bar{a}_{21}}{(\bar{a}_{22})^2} \right) \frac{\partial u}{\partial r} \frac{\partial \psi}{\partial r} + \int_{\mathcal{O}^-} B\nabla u \nabla\psi.
\end{aligned}$$

Since A and B are symmetric, we have $\bar{a}_{12} = \bar{a}_{21}$ and $\bar{b}_{12} = \bar{b}_{21}$. Using matrix notation, we can simplify the above equation as

$$\int_{\mathcal{O}_U} \left(\frac{\det \bar{A}}{\bar{a}_{22}} \right) \frac{\partial v}{\partial r} \frac{\partial \psi}{\partial r} + k'(u) v\psi + v\psi + \int_{\mathcal{O}^-} A\nabla u \nabla\psi + k'(u) v\psi + v\psi$$

$$
= \int_{\mathcal{O}_U} \left(\bar{B} \begin{bmatrix} 1 \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \end{bmatrix} \frac{\partial u}{\partial r} \frac{\partial \psi}{\partial r} \right) + \int_{\mathcal{O}^-} B \nabla u \nabla \psi.
$$

Taking average using the properties of the unfolding operator to get

$$
\begin{aligned}
&\int_{\mathcal{O}^+} a_0 \frac{\partial v}{\partial r} \frac{\partial \psi}{\partial r} + |Y(r)| \left(k'(u)v + v\right) \psi + \int_{\mathcal{O}^-} A \nabla u \nabla \psi + (k'(u)v + v\psi) \\
&= \int_{\mathcal{O}^+} b_0 \frac{\partial u}{\partial r} \frac{\partial \psi}{\partial r} + \int_{\mathcal{O}^-} B \nabla u \nabla \psi,
\end{aligned}
$$

where the coefficients a_0 and b_0 are given by

$$
a_0 = \int_{Y(r)} \left(\frac{\det \bar{A}}{\bar{a}_{22}} \right) d\tau \quad \text{and} \quad b_0 = \int_{Y(r)} \left(\bar{B} \begin{bmatrix} 1 \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \end{bmatrix} \right) d\tau.
$$

This completes the proof. □

3 Homogenization in Domains with Lower Dimensional Oscillations

In this section, we will discuss the homogenization result for a semi-linear partial differential equation (PDE) and its associated optimal control problem in an n-dimensional domain with oscillating boundary. The oscillations occur in m directions, where m ranges from 1 to $n - 1$.

3.1 Domain Description

Let $x = (x', x'') \in \mathbb{R}^n$ where $x' = (x_1, x_2, \ldots x_m)$ and $x'' = (x_{m+1}, x_{m+2}, \ldots x_n)$ with $1 < m < n$. Define

$$
\Omega^+ = (0, 1)^n, \quad \text{and} \quad Y^* = \prod_{i=1}^{m} (a_i, b_i) \times (0, 1)^{n-m}
$$

with $0 < a_i < b_i < 1$ for all $i = 1, 2, 3, \ldots, m$. Let Λ be a connected open subset of Y^* with Lipschitz boundary as our reference cell. Now the upper oscillating part Ω_ε^+ is given by

$$
\Omega_\varepsilon^+ = \left\{ (x', x'') \in \Omega^+ : \left(\left\{ \frac{x'}{\varepsilon} \right\}, x'' \right) \in \Lambda \right\},
$$

where $\left\{ \frac{x'}{\varepsilon} \right\}$ denotes the fractional part of $\frac{x'}{\varepsilon}$. The lower fixed part is given by

$$
\Omega^- = (0, 1)^{n-1} \times (-1, 0).
$$

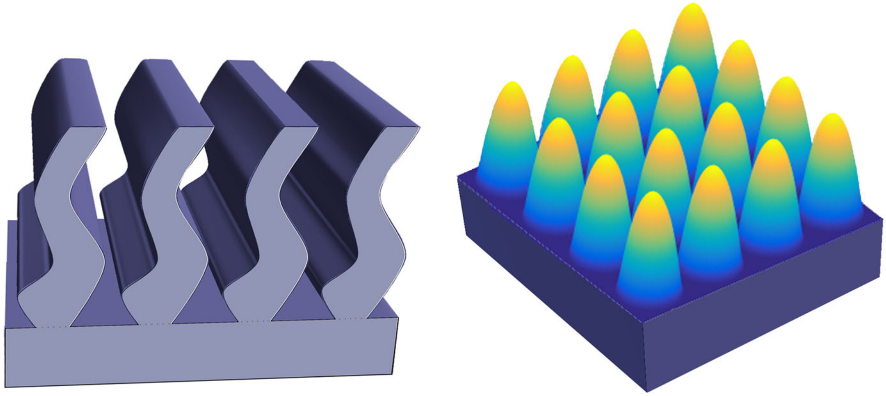


Fig. 4 3 Dimensional oscillating domains with $m = 1$ and $m = 2$

The oscillating domain Ω_ε and limit domain Ω are defined as

$$\Omega_\varepsilon = \text{int}\left(\overline{\Omega_\varepsilon^+ \cup \Omega^-}\right) \text{ and } \Omega = \text{int}\left(\overline{\Omega^+ \cup \Omega^-}\right).$$

Here Ω_ε^+ is the upper oscillating part, Ω^- is the lower fixed part, Ω_ε is the oscillating domain and Ω is the limit domain. Sample figures are given in Fig. 4. It is important to note that as per the definition of Ω_ε, the upper part Ω_ε^+ exhibits periodic oscillations. These oscillations involve a periodic arrangement of the reference cell Λ, which is scaled by ε in the x' variable and arranged in the x' direction with a period of ε. Also Γ_a, Γ_b are upper and lower boundaries of Ω and Γ_0 is the interface.

For $x'' \in (0,1)^{n-m}$, define $Y(x'') = \left\{y \in (0,1)^m : (y, x'') \in \Lambda\right\}$ where $\left|Y(x'')\right|$ denote the m dimensional Lebesgue measure of $Y(x'')$. We assume the following properties on Λ:

1. The set $Y(x'')$ is connected for all $x'' \in (0,1)^{n-m}$,
2. There exists $\rho > 0$ such that $0 < \rho \leq |Y(x'')| < 1$ for all $x'' \in (0,1)^{n-m}$,
3. The boundary part $\partial\Lambda \cap \left((0,1)^{n-1} \times \{0\}\right)$ is connected and have positive $n-1$ dimensional Lebesgue measure.

3.2 Periodic Unfolding Operator

We have already introduced the domain Ω_ε with a highly oscillating boundary. First, we will define the unfolded domain Ω_U in which the unfolded functions are defined. The unfolded domain Ω_U is defined as follows:

$$\Omega_U = \left\{(x, y) \mid x = (x', x'') \in \Omega^+, y \in Y(x'') \subset \mathbb{R}^m\right\}.$$

Let $\mathcal{G} = \{(x'', y) \mid x'' \in (0,1)^{n-m}, y \in Y(x'')\}$, then, one can write, $\Omega_U = (0,1)^m \times \mathcal{G}$. Let $\phi^\varepsilon : \Omega_U \to \Omega_\varepsilon^+$ be defined as $\phi^\varepsilon(x, y) = \left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y, x''\right)$. The $\varepsilon-$ unfolding of a function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega_U \to \mathbb{R}$.

The operator which maps every function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. We denote the unfolding operator by T^ε, that is,

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon(u) : \Omega_U \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon(u)(x, y) = u\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y, x''\right).$$

If $V \subset \mathbb{R}^N$ containing Ω_ε^+ and u is a real-valued function on V, $T^\varepsilon(u)$ means, that is T^ε acting on the restriction of u to Ω_ε^+. Some important properties of the unfolding operator are stated below. For each $\varepsilon > 0$,

1. T^ε is linear. Further, if $u,\ v : \Omega_\varepsilon^+ \to \mathbb{R}$, then, $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.
2. Let $u \in L^1(\Omega_\varepsilon^+)$. then,

$$\int_{\Omega_U} T^\varepsilon(u) = \int_{\Omega_\varepsilon^+} u.$$

3. Let $u \in L^2(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(\Omega_U)$ and $\|T^\varepsilon u\|_{L^2(\Omega_U)} = \|u\|_{L^2(\Omega_\varepsilon^+)}$.
4. Let $u \in H^1(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2((0, 1)^m; H^1(\mathcal{G}))$. Moreover,

$$\nabla_{x''} T^\varepsilon u = T^\varepsilon \nabla_{x''} u \quad \text{and} \quad \nabla_y T^\varepsilon u = \varepsilon T^\varepsilon \nabla_{x'} u.$$

5. Let $u \in L^2(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \to u$ strongly in $L^2(\Omega_U)$. More generally, let $u_\varepsilon \to u$ strongly in $L^2(\Omega^+)$. Then, $T^\varepsilon u_\varepsilon \to u$ strongly in $L^2(\Omega_U)$.
6. Let, for every ε, $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega_U)$. then,

$$\tilde{u}_\varepsilon \rightharpoonup \int_{Y(x'')} u(x, y)dy \quad \text{weakly in } \ L^2(\Omega^+).$$

7. Let, for every $\varepsilon > 0$, $u_\varepsilon \in H^1(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0, 1)^m; H^1(\mathcal{G}))$. Then,

$$\widetilde{u}_\varepsilon \rightharpoonup \int_{Y(x'')} u(x, y)dy \text{ weakly in } L^2(\Omega^+) \text{ and}$$
$$\widetilde{\nabla_{x''} u_\varepsilon} \rightharpoonup \int_{Y(x'')} \nabla_{x''} u dy \quad \text{weakly in } \ L^2(\Omega^+)^{n-m}.$$

where $\widetilde{u}_\varepsilon$ denotes the extension by 0 of u_ε to Ω^+. This notation is used throughout the article.

3.3 Homogenization

Let $A = [a_{ij}]_{n\times n}$ be an $n \times n$ symmetric matrix, where the entries $a_{ij} \in L^\infty(\Omega)$. Also A is uniformly elliptic and bounded in Ω, that is, there exists $\alpha, \beta > 0$ such that

$$\langle A(x)\lambda, \lambda\rangle \geq \alpha|\lambda|^2 \quad \text{and} \quad |A(x)\lambda| \leq \beta|\lambda|$$

for all $\lambda \in \mathbb{R}^n$ and $a.e$ in Ω. Let A_1, A_2, A_3, A_4 be sub-matrices of A such that

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where the orders of the sub-matrices are as follows:

$$A_1 : m \times m, \quad A_2 : m \times (n-m), \quad A_3 : (n-m) \times m, \quad A_4 : (n-m) \times (n-m).$$

Consider the following problem in Ω_ε:

$$\begin{cases} -\text{div}(A\nabla u_\varepsilon) + k(u_\varepsilon) + u_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ \qquad\qquad A\nabla u \cdot \nu_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon. \end{cases}$$

Here $f \in L^2(\Omega)$ is a given function, ν^ε is the outward unit normal vector, and k is as defined in the earlier section. The corresponding variational formulation is

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} A\nabla u_\varepsilon \nabla\phi + k(u_\varepsilon)\phi + u_\varepsilon\phi = \int_{\Omega_\varepsilon} f\phi, \text{ for all } \phi \in H^1(\Omega_\varepsilon). \end{cases} \tag{21}$$

The existence and uniqueness of u_ε is guaranteed by the Browder–Minty theorem. We want to study the asymptotic behavior of u_ε as $\varepsilon \to 0$. Let us look at the limit problem.

Limit problem: Consider the Hilbert space

$$W(\Omega) = \left\{\psi \in L^2(\Omega) : \nabla_{x''}\psi \in L^2(\Omega)^{n-m}, \psi|_{\Omega^-} \in H^1(\Omega^-)\right\}$$

with inner product

$$\langle\phi, \psi\rangle_{W(\Omega)} = \langle\phi, \psi\rangle_{L^2(\Omega^+)} + \langle\nabla_{x''}\phi, \nabla_{x''}\psi\rangle_{L^2(\Omega^+)} + \langle\phi, \psi\rangle_{H^1(\Omega^-)}.$$

We define the limit problem as follows: Given $f \in L^2(\Omega)$, find $u \in W(\Omega)$ such that

$$\begin{aligned} &\int_{\Omega^+} A_0\nabla_{x''}u\nabla_{x''}\psi + |Y(x'')|k(u)\psi + u\psi + \int_{\Omega^-} A\nabla u\nabla\psi + k(u)\psi + u\psi \\ &= \int_{\Omega^+} |Y(x'')|f\psi + \int_{\Omega^-} f\psi, \end{aligned} \tag{22}$$

for all $\psi \in W(\Omega)$, where

$$A_0(x) = A_0(x'') = |Y(x'')| \left(\left[-A_3 A_1^{-1} \; I \right] A \left[-A_3 A_1^{-1} \; I \right]^t \right)$$

Corresponding strong form is

$$\begin{cases} -\mathrm{div}_{x''}(A_0 \nabla_{x''} u^+) + |Y(x'')|(k(u^+) + u^+) = |Y(x'')| f & \text{in } \Omega^+, \\ -\mathrm{div}(A \nabla u^-) + k(u^-) + u^- = f & \text{in } \Omega^-, \\ A_0 \nabla_{x''} u^+ \cdot \nu = 0 & \text{on } \Gamma_a, \\ A \nabla u^- \cdot \nu = 0 & \text{on } \Gamma_b, \\ u^+ = u^-, \quad A_0 \nabla_{x''} u^+ \cdot \nu = A \nabla u^- \cdot \nu & \text{on } \Gamma_0. \end{cases}$$

Since A is symmetric and coercive, and k is monotone, by Browder–Minty theorem, (22) has a unique solution.

Theorem 6 *Let u_ε, u be the unique solutions of (21) and (22) respectively. Then, we have the following convergences*

$$\begin{aligned} &\widetilde{u_\varepsilon} \rightharpoonup u \text{ weakly in } L^2(\Omega), \\ &\widetilde{\nabla_{x''} u_\varepsilon} \rightharpoonup \nabla_{x''} u \text{ weakly in } L^2(\Omega^+)^{n-m}, \\ &\widetilde{\nabla_{x'} u_\varepsilon} \rightharpoonup (-A_1^{-1} A_2) \nabla_{x''} u \text{ weakly in } L^2(\Omega^+)^m, \\ &\widetilde{k(u_\varepsilon)} \rightharpoonup |Y(x'')| k(u) \text{ weakly in } L^2(\Omega^+), \\ &u_\varepsilon \longrightarrow u \text{ weakly in } H^1(\Omega^-). \end{aligned}$$

Proof We are dividing the proof into several steps.
Step 1: (Convergences) Since $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \|f\|_{L^2(\Omega)}$, by using the properties of unfolding operator defined in Sect. 3.2 we have $\{T^\varepsilon(u_\varepsilon)\}$ is bounded in $L^2((0,1)^m; H^1(\mathcal{G}))$. Also $\{u_\varepsilon\}$ is bounded in $H^1(\Omega^-)$. Hence from weak compactness, there exist $u^+ \in L^2(\Omega_U)$, $u^- \in H^1(\Omega^-)$, $P_1 \in L^2(\Omega_U)^m$ and $P_2 \in L^2(\Omega_U)^{n-m}$ such that

$$\begin{aligned} &T^\varepsilon u_\varepsilon \rightharpoonup u^+ \text{ weakly in } L^2(\Omega_U), \\ &T^\varepsilon(\nabla_{x'} u_\varepsilon) \rightharpoonup P_1 \text{ weakly in } L^2(\Omega_U)^m, \\ &T^\varepsilon(\nabla_{x''} u_\varepsilon) \rightharpoonup P_2 \text{ weakly in } L^2(\Omega_U)^{n-m}, \\ &T^\varepsilon(k(u_\varepsilon)) \rightharpoonup \zeta \text{ weakly in } L^2(\Omega_U), \\ &u_\varepsilon \longrightarrow u^- \text{ weakly in } H^1(\Omega^-). \end{aligned} \tag{23}$$

From the properties of unfolding, it is easy to see that

$$P_2 = \nabla_{x''} u^+.$$

Using the similar properties, we get

$$\nabla_y T^\varepsilon(u_\varepsilon) \rightharpoonup \nabla_y u \text{ weakly in } L^2(\Omega_U)^m.$$

But

$$\nabla_y T^\varepsilon(u_\varepsilon) = \varepsilon T^\varepsilon \nabla_{x'} u \rightharpoonup 0 \text{ weakly in } L^2(\Omega_U),$$

which implies u^+ is independent of y. Next step is to identify P_1. For $\phi \in D(\Omega^+)$ and $\psi \in C^\infty([0,1]^m)$, define $\phi^\varepsilon = \varepsilon\phi(x)\psi\left(\left\{\frac{x'}{\varepsilon}\right\}\right)$. Then

$$\begin{aligned} &T^\varepsilon(\phi^\varepsilon) = \varepsilon T^\varepsilon(\phi)\psi(y), \quad T^\varepsilon(\nabla_{x''}\phi^\varepsilon) = \varepsilon T^\varepsilon(\nabla_{x''}\phi)\,\psi(y) \quad \text{and} \\ &T^\varepsilon\left(\nabla_{x'}\phi^\varepsilon\right) = \varepsilon T^\varepsilon(\nabla_{x'}\phi) + T^\varepsilon(\phi)\,\nabla_y\psi(y). \end{aligned} \tag{24}$$

Use ϕ^ε as a test function in (21) to get

$$\int_{\Omega_\varepsilon^+} A\nabla u_\varepsilon \nabla\phi_\varepsilon + k(u_\varepsilon)\phi_\varepsilon + u_\varepsilon\phi_\varepsilon = \int_{\Omega_\varepsilon^+} f\phi_\varepsilon.$$

Apply the unfolding operator and pass to the limit using (23) and (24) to get

$$\int_{\Omega_U} A\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}\begin{bmatrix} \phi\nabla_y\psi \\ 0 \end{bmatrix} = 0.$$

Since ϕ and ψ are arbitrary, $A_1P_1 + A_2P_2 = 0$, which implies

$$P_1 = -A_1^{-1}A_2P_2 = -A_1^{-1}A_2\nabla_{x''}u^+. \tag{25}$$

Step 2: (Interface Condition) In this step, we are going to prove that $u^+ = u^-$ on Γ. By the continuity of the trace operator and using properties of the unfolding operator, we get

$$\begin{aligned} \int_\Gamma u^+\phi &= \lim_{\varepsilon\to 0}\int_\Gamma \left(T^\varepsilon(u_\varepsilon)\right)\big|_{x_n=0} T_0^\varepsilon(\phi) = \lim_{\varepsilon\to 0}\int_\Gamma \left(T_0^\varepsilon(u_\varepsilon|_{\Omega^+})\right)\big|_{x_n=0} T_0^\varepsilon(\phi) \\ &= \lim_{\varepsilon\to 0}\int_\Gamma \left(T_0^\varepsilon(u_\varepsilon|_{\Omega^-})\right)\big|_{x_n=0} T_0^\varepsilon(\phi) = \int_\Gamma u^-\phi, \end{aligned}$$

for any $\phi \in C_c^\infty(\Gamma)$. Hence, we have $u^+ = u^-$ on Γ. Define

$$u = \chi_{\mathcal{O}^+}u^+ + \chi_{\mathcal{O}^-}u^-.$$

Since $\nabla_{x''}u^+ \in L^2(\Omega^+)^{n-m}$ and $u^- \in H^1(\Omega^-)$, the interface condition gives $u \in W(\Omega)$.

Step 3: (Identifying ζ) As in the previous section, we need to identify ζ. The computation is delicate because it involves higher order matrices, and we are using the Browder–Minty method to perform it. Let $\phi \in C^1(\bar{\Omega})$. Consider the integral

$$I_\varepsilon = \int_{\Omega_\varepsilon^+} A \begin{bmatrix} \nabla_{x'} u_\varepsilon - \left(-A_1^{-1} A_2\right) \nabla_{x''} u \\ \nabla_{x''} u_\varepsilon - \nabla_{x''}\phi \end{bmatrix} \begin{bmatrix} \nabla_{x'} u_\varepsilon - \left(-A_1^{-1} A_2\right) \nabla_{x''} u \\ \nabla_{x''} u_\varepsilon - \nabla_{x''}\phi \end{bmatrix}$$
$$+ \int_{\Omega_\varepsilon^+} (k(u_\varepsilon) - k(\phi)(u_\varepsilon - \phi)) + (u_\varepsilon - \phi)^2$$
$$+ \int_{\Omega^-} A(\nabla u_\varepsilon - \nabla\phi)(\nabla u_\varepsilon - \nabla\phi) + (k(u_\varepsilon) - k(\phi)(u_\varepsilon - \phi)) + (u_\varepsilon - \phi)^2.$$

Expand and rearrange to get

$$I_\varepsilon = \int_{\Omega_\varepsilon} A\nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2 + \int_{\Omega_\varepsilon^+} -A \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \nabla_{x''} u_\varepsilon \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''}\phi \end{bmatrix}$$
$$+ \int_{\Omega_\varepsilon^+} -A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''}\phi \end{bmatrix} \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \nabla_{x''} u_\varepsilon \end{bmatrix}$$
$$+ \int_{\Omega_\varepsilon^+} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''}\phi \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''}\phi \end{bmatrix}$$
$$+ \int_{\Omega_\varepsilon^+} -k(u_\varepsilon)\phi - k(\phi)u_\varepsilon + k(\phi)\phi - 2u_\varepsilon\phi + \phi^2$$
$$+ \int_{\Omega^-} -A\nabla u_\varepsilon \nabla\phi - A\nabla\phi\nabla u_\varepsilon + A\nabla\phi\nabla\phi$$
$$+ \int_{\Omega^-} -k(u_\varepsilon)\phi - k(\phi)u_\varepsilon + k(\phi)\phi - 2u_\varepsilon\phi + \phi^2.$$

Now we have to pass the limit as $\varepsilon \to 0$. Using (23) pass to the limit in the variational form (21) to get

$$\int_{\Omega_U} f\phi + \int_{\Omega^-} f\phi = \int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \nabla\phi + \zeta\phi + u\phi + \int_{\Omega^-} A\nabla u \nabla\phi + k(u)\phi + u\phi$$
$$= \int_{\Omega_U} \left(A_4 - A_3 A_1^{-1} A_2\right) \nabla_{x''} u \nabla_{x''}\phi + \zeta\phi + u\phi$$
$$+ \int_{\Omega^-} A\nabla u \nabla\phi + k(u)\phi + u\phi.$$

By density of $C^1(\bar{\Omega})$ in $W(\Omega)$, the above equality holds for all $\phi \in W(\Omega)$. Put $\phi = u$ to get

$$\int_{\Omega_U} fu + \int_{\Omega^-} fu = \int_{\Omega_U} \left(A_4 - A_3 A_1^{-1} A_2\right) \nabla_{x''} u \nabla_{x''} u + \zeta u + u^2$$
$$+ \int_{\Omega^-} A\nabla u \nabla u + k(u)u + u^2$$

$$= \int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} + \zeta u + u^2$$
$$+ \int_{\Omega^-} A \nabla u \nabla u + k(u) u + u^2.$$

On the other hand, using the energy equality we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} A \nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon) u_\varepsilon + u_\varepsilon^2 &= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f u_\varepsilon = \int_{\Omega_U} f u + \int_{\Omega^-} f u \\ &= \int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} + \zeta u + u^2 \\ &\quad + \int_{\Omega^-} A \nabla u \nabla u + k(u) u + u^2. \end{aligned} \tag{26}$$

Now pass to the limit as $\varepsilon \to 0$ in I_ε using (23) and (26) to get

$$\begin{aligned} \lim_{\varepsilon \to 0} I_\varepsilon = &\int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \\ &- \int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} \\ &- \int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \\ &+ \int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} \\ &+ \int_{\Omega_U} \zeta u - \zeta \phi - k(\phi) u + k(\phi) \phi + u^2 - 2u\phi + \phi^2 \\ &+ \int_{\Omega^-} A \nabla u \nabla u - A \nabla u \nabla \phi - A \nabla \phi \nabla u + A \nabla \phi \nabla \phi \\ &+ \int_{\Omega^-} k(u) u - k(u) \phi - k(\phi) u + k(\phi) \phi + u^2 - 2u\phi + \phi^2. \end{aligned}$$

By properly factoring, we can obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} I_\varepsilon = &\int_{\Omega_U} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u + A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u - \nabla_{x''} \phi \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u + A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u - \nabla_{x''} \phi \end{bmatrix} \\ &+ \int_{\Omega_U} (\zeta - k(u))(u - \phi) + (u - \phi)^2 \\ &+ \int_{\Omega^-} A(\nabla u - \nabla \phi)(\nabla u - \nabla \phi) + (k(u) - k(\phi))(u - \phi) + (u - \phi)^2 \\ = &\int_{\Omega_U} A_4 (\nabla_{x''} u - \nabla_{x''} \phi)(\nabla_{x''} u - \nabla_{x''} \phi) + (\zeta - k(\phi))(u - \phi) + (u - \phi)^2 \end{aligned}$$

$$+\int_{\Omega^-} A(\nabla u-\nabla\phi)(\nabla u-\nabla\phi)+(k(u)-k(\phi))(u-\phi)+(u-\phi)^2.$$

From the monotonicity of k, we have $I^\varepsilon \geq 0$ for all ε, which implies

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon =& \int_{\Omega_U} A_4(\nabla_{x''}u-\nabla_{x''}\phi)(\nabla_{x''}u-\nabla_{x''}\phi)+(\zeta-k(\phi))(u-\phi)+(u-\phi)^2\\ &+\int_{\Omega^-} A(\nabla u-\nabla\phi)(\nabla u-\nabla\phi)+(k(u)-k(\phi))(u-\phi)+(u-\phi)^2 \geq 0.\end{aligned}$$

Choose $\phi = u+\lambda\psi$, $\psi \in C^\infty\left(\overline{\Omega}\right)$, $\lambda > 0$ to get

$$\begin{aligned}&\int_{\Omega_U} \lambda A_4\nabla_{x''}\psi\nabla_{x''}\psi+(\zeta-k(\phi-\lambda\psi))\psi+\lambda\psi^2\\ &\quad+\int_{\Omega^-} \lambda A\nabla\psi\nabla\psi+(k(u)-k(u-\lambda\psi))\psi+\lambda\psi^2 \geq 0.\end{aligned}$$

As $\lambda \to 0$,

$$\int_{\Omega_U} (\zeta-k(u))\,\psi \geq 0 \text{ for all } \psi \in C^1\left(\overline{\Omega}\right).$$

Hence,

$$\int_{Y(x'')} \zeta\, dy = |Y(x'')|k(u). \tag{27}$$

We have evaluated all the unknowns in (23). Hence using properties of the unfolding operator, we can deduce the following convergence from (23) using (25),(27), and interface condition.

$$\begin{aligned}&\widetilde{u_\varepsilon} \rightharpoonup |Y(x'')|u \text{ weakly in } L^2(\Omega),\\ &\widetilde{\nabla_{x''}u_\varepsilon} \rightharpoonup |Y(x'')|\nabla_{x''}u \text{ weakly in } L^2(\Omega^+)^{n-m},\\ &\widetilde{\nabla_{x'}u_\varepsilon} \rightharpoonup |Y(x'')|(-A_1^{-1}A_2)\nabla_{x''}u \text{ weakly in } L^2(\Omega^+)^m,\\ &\widetilde{k(u_\varepsilon)} \rightharpoonup |Y(x'')|k(u) \text{ weakly in } L^2(\Omega^+),\\ &u_\varepsilon \longrightarrow u \text{ weakly in } H^1(\Omega^-).\end{aligned}$$

Hence we got the required convergence. Now we need to prove that u is actually the solution of the limit problem (22).

Step 4: (Limit Problem) Use $\psi \in C^\infty(\bar{\Omega})$ as test function in (21). Apply unfolding operator and passing to the limit using (23), we obtain

$$\int_{\Omega_U} A\begin{bmatrix}P_1\\P_2\end{bmatrix}\nabla\psi+\zeta\psi+u\psi+\int_{\Omega^-} A\nabla u\nabla\psi+k(u)\psi+u\psi=\int_{\Omega_U} f\psi+\int_{\Omega^-} f\psi.$$

Simplify using values of (25),

$$\int_{\Omega_U} (A_3 P_1 + A_4 P_2)\, \nabla_{x''}\psi + \zeta\psi + u\psi + \int_{\Omega^-} A\nabla u \nabla\psi + k(u)\psi + u\psi$$
$$= \int_{\Omega_U} f\psi + \int_{\Omega^-} f\psi.$$

Substitute values of P_1 and P_2,

$$\int_{\Omega_U} \left(-A_3 A_1^{-1} A_2 + A_4\right) \nabla_{x''}u \nabla_{x''}\psi + \zeta\psi + u\psi + \int_{\Omega^-} A\nabla u \nabla\psi + k(u)\psi + u\psi$$
$$= \int_{\Omega_U} f\psi + \int_{\Omega^-} f\psi.$$

Average out using (27) and properties of the unfolding operator to get

$$\int_{\Omega^+} A_0 \nabla_{x''}u \nabla_{x''}\psi + |Y(x'')| k(u)\psi + u\psi + \int_{\Omega^-} A\nabla u \nabla\psi + k(u)\psi + u\psi$$
$$= \int_{\Omega^+} |Y(x'')| f\psi + \int_{\Omega^-} f\psi,$$

where the coefficient matrix A_0 is given by

$$A_0 = \int_{Y(x'')} \left(-A_3 A_1^{-1} A_2 + A_4\right) dy = |Y(x")| \left(-A_3 A_1^{-1} A_2 + A_4\right).$$

To prove the existence and uniqueness of solution for the variational form, a major challenge is to show that A_0 is coercive. Interestingly, we could obtain a different matrix expression for A_0 which directly implies its coercivity due to the coercivity of A. Using the symmetric property of A we can rewrite A_0 as

$$A_0 = |Y(x'')| \left(\left[-A_3 A_1^{-1}\ I\right] A \left[-A_3 A_1^{-1}\ I\right]^t\right).$$

By density of $C^\infty(\bar{\Omega})$, in $W(\Omega)$, we get that u satisfies the limit problem (22). Hence the proof of Theorem 6 is done. □

We will prove the corresponding results in the following theorem.

Theorem 7 (Corrector results) *Let u_ε, u be the unique solutions of* (21) *and* (22) *respectively. Then, we have the following convergences*

$$
\begin{aligned}
\widetilde{u_\varepsilon} - \chi_{\Omega_\varepsilon} u &\longrightarrow 0 \ \textit{strongly in } L^2(\Omega), \\
\widetilde{\nabla_{x''} u_\varepsilon} - \chi_{\Omega_\varepsilon} \nabla_{x''} u &\longrightarrow 0 \ \textit{strongly in } L^2(\Omega^+)^{n-m}, \\
\widetilde{\nabla_{x'} u_\varepsilon} - \chi_{\Omega_\varepsilon} \left(-A_1^{-1} A_2\right) \nabla_{x''} u &\longrightarrow 0 \ \textit{strongly in } L^2(\Omega^+)^m, \\
u_\varepsilon - u &\longrightarrow 0 \ \textit{strongly in } H^1(\Omega^-).
\end{aligned}
$$

Proof Consider

$$
\begin{aligned}
J_\varepsilon = &\int_{\Omega_\varepsilon^+} A \begin{bmatrix} \nabla_{x'} u_\varepsilon - \left(-A_1^{-1} A_2 \nabla_{x''} u\right) \\ \nabla_{x''} u_\varepsilon - \nabla_{x''} u \end{bmatrix} \begin{bmatrix} \nabla_{x'} u_\varepsilon - \left(-A_1^{-1} A_2 \nabla_{x''} u\right) \\ \nabla_{x''} u_\varepsilon - \nabla_{x''} u \end{bmatrix} \\
&+ \int_{\Omega_\varepsilon^+} \left(k(u_\varepsilon) - k(u)\right)(u_\varepsilon - u) + (u_\varepsilon - u)^2 \\
&+ \int_{\Omega^-} A(\nabla u_\varepsilon - \nabla u)(\nabla u_\varepsilon - \nabla u) + \left(k(u_\varepsilon) - k(u)\right)(u_\varepsilon - u) + (u_\varepsilon - u)^2.
\end{aligned}
$$

Expand and rearrange to get

$$
J_\varepsilon = J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3 + J_\varepsilon^4,
$$

where

$$
\begin{aligned}
J_\varepsilon^1 &= \int_{\Omega_\varepsilon} A \nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon) u_\varepsilon + u_\varepsilon^2, \\
J_\varepsilon^2 &= \int_{\Omega_\varepsilon^+} -A \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \nabla_{x''} u_\varepsilon \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} + \int_{\Omega_\varepsilon^+} -A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \nabla_{x''} u_\varepsilon \end{bmatrix} \\
&\quad + \int_{\Omega_\varepsilon^+} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix}, \\
J_\varepsilon^3 &= \int_{\Omega_\varepsilon^+} -k(u_\varepsilon) u - k(u) u_\varepsilon + k(u) u - 2 u_\varepsilon u + u^2, \\
J_\varepsilon^4 &= \int_{\Omega^-} -A \nabla u_\varepsilon \nabla u - A \nabla u \nabla u_\varepsilon + A \nabla u \nabla u \\
&\quad + \int_{\Omega^-} -k(u_\varepsilon) u - k(u) u_\varepsilon + k(u) u - 2 u_\varepsilon u + u^2.
\end{aligned}
$$

On applying the unfolding operator and passing to the limit as $\varepsilon \to 0$, we get

$$
\begin{aligned}
\lim_{\varepsilon \to 0} J_\varepsilon^2 &= \int_{\Omega_U} \left(A_3 A_1^{-1} A_2 - A_4\right) \nabla_{x''} u \nabla_{x''} u \\
&= \int_{\Omega^+} -A_0 \nabla_{x''} u \nabla_{x''} u, \\
\lim_{\varepsilon \to 0} J_\varepsilon^3 &= \int_{\Omega_U} -\zeta u - u^2 = \int_{\Omega^+} -|Y(x'')| \left(k(u) u + u^2\right),
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0} J_\varepsilon^4 &= \int_{\Omega^-} -A\nabla u\nabla u - k(u)u - u^2,\\
\lim_{\varepsilon\to 0} J_\varepsilon^1 &= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A\nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2\\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f u_\varepsilon = \int_{\Omega_U} fu + \int_{\Omega^-} fu\\
&= \int_{\Omega^+} A_0\nabla_{x''}u\nabla_{x''}u + |Y(x'')|\left(k(u)u + u^2\right) + \int_{\Omega^-} A\nabla u\nabla u + k(u)u + u^2\\
&= -\left(\lim_{\varepsilon\to 0} J_2^\varepsilon + \lim_{\varepsilon\to 0} J_3^\varepsilon + \lim_{\varepsilon\to 0} J_4^\varepsilon\right).
\end{aligned}
$$

This implies that

$$\lim_{\varepsilon\to 0} I_\varepsilon = 0.$$

Then coercivity of A and monotonicity of k completes the proof of Theorem 7. □

3.4 Optimal Control

Define A as in Sect. 3.3. Also define

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

in a similar way as we have defined A. Let $\omega \subset\subset \Omega^-$ be an open set and admissible control set is $L^2(\omega)$. Now consider the following optimal control problem: Minimize

$$J_\varepsilon(u,\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B\nabla u\nabla u + \frac{\beta}{2}\int_{\Omega_\varepsilon} \chi_\omega |\theta|^2, \tag{28}$$

where (u, θ) satisfies the following system

$$\begin{cases} -\mathrm{div}(A\nabla u) + k(u) + u = f + \chi_\omega\theta \text{ in } \Omega_\varepsilon,\\ A\nabla u\cdot \nu_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon,\end{cases}$$

where $f \in L^2(\Omega)$. From the semi-linear optimal control theory, we have the existence and uniqueness of the optimal solution $(u_\varepsilon, \theta_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(\omega)$ (see [6, 15, 55]).

We aim to study the asymptotic behavior of $(u_\varepsilon, \theta_\varepsilon)$ as $\varepsilon \to 0$. Let $(u_\varepsilon, \theta_\varepsilon)$ be the unique solution of (28). Then $(u_\varepsilon, v_\varepsilon)$ will satisfy the following optimality system.

$$\begin{cases} -\operatorname{div}(A\nabla u_\varepsilon) + k(u_\varepsilon) + u_\varepsilon = f + \chi_\omega \theta_\varepsilon \text{ in } \Omega_\varepsilon, \\ -\operatorname{div}(A\nabla v_\varepsilon) + k'(u_\varepsilon)v_\varepsilon + v_\varepsilon = -\operatorname{div}(B\nabla u_\varepsilon) \text{ in } \Omega_\varepsilon, \\ A\nabla u_\varepsilon \cdot \nu_\varepsilon = 0, \quad A\nabla v_\varepsilon \cdot \nu_\varepsilon = B\nabla u_\varepsilon \text{ on } \partial\Omega_\varepsilon, \\ \theta_\varepsilon = \dfrac{1}{\beta} v_\varepsilon. \end{cases}$$

Corresponding variational form is: Given $f \in L^2(\Omega)$, find $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ such that

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon} A\nabla u_\varepsilon \nabla \psi + (k(u_\varepsilon) + u_\varepsilon)\psi = \int_{\Omega_\varepsilon} (f + \chi_\omega \theta_\varepsilon)\psi, \\ \displaystyle\int_{\Omega_\varepsilon} A\nabla v_\varepsilon \nabla \phi + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)\phi = \int_{\Omega_\varepsilon} B\nabla u_\varepsilon \nabla \phi, \end{cases} \tag{29}$$

for all $(\psi, \phi) \in H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ with

$$\theta_\varepsilon = -\frac{1}{\beta}\chi_\omega v_\varepsilon.$$

We want to study the asymptotic behavior of $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0$. We now describe the limit optimal control problem which we will be the homogenized problem (Theorem 9).

For the limit optimal control problem, the admissible control set is again $L^2(\omega)$. The limit optimal control problem is given as follows: Minimize

$$J(u, \theta) = \frac{1}{2}\int_{\Omega^+} B_0 \nabla_{x''} u^+ \nabla_{x''} u^+ + \frac{1}{2}\int_{\Omega^-} B\nabla u^- \nabla u^- + \frac{1}{2}\int_{\Omega^-} \chi_\omega |\theta|^2,$$

where (u, θ) satisfies the following system

$$\begin{cases} -\operatorname{div}_{x''}(A_0 \nabla_{x''} u^+) + |Y(x'')| k(u^+) + u^+ = |Y(x'')| f & \text{in } \Omega^+, \\ -\operatorname{div}(A\nabla u^-) + k(u^-) + u^- = f + \chi_\omega \theta & \text{in } \Omega^-, \\ A_0 \nabla_{x''} u^+ \cdot \nu = 0 & \text{on } \Gamma_a, \\ A\nabla u^- \cdot \nu = 0 & \text{on } \Gamma_b, \\ A_0 \nabla_{x''} u^+ \cdot \nu - A\nabla u^- \cdot \nu = 0 & \text{on } \Gamma_0. \end{cases}$$

where the coefficient matrix A_0 and B_0 are given by

$$A_0 = \left|Y(x'')\right| \left(\left[-A_3 A_1^{-1} \; I\right] A \left[-A_3 A_1^{-1} \; I\right]^t \right) \quad \text{and}$$
$$B_0 = \left|Y(x'')\right| \left(\left[-A_3 A_1^{-1} \; I\right] B \left[-A_3 A_1^{-1} \; I\right]^t \right).$$

The definition A_0 and B_0 implies the coerciveness of A_0 and B_0. We already have monotonicity of k, then by semi-linear optimal control theory, we have the existence and uniqueness of the optimal solution $(\bar{u}, \theta) \in W(\Omega) \times L^2(\omega)$ (see [6, 15]).

Again from the well-known theory for semi-linear optimal control problems (see [15, 55]) we can write the optimality system corresponding to the limit optimal control problem as follows:

$$\begin{cases} -\mathrm{div}_{x''}(A_0 \nabla_{x''} u^+) + |Y(x'')| k(u^+) + u^+ = |Y(x'')| f & \text{in } \Omega^+, \\ -\mathrm{div}(A_0 \nabla_{x''} v^+) + k'(u^+) v^+ + v^+ = -\mathrm{div}(B_0 \nabla_{x''} u^+) & \text{in } \Omega^+, \\ -\mathrm{div}(A \nabla u^-) + k(u^-) + u^- = f + \chi_\omega \theta & \text{in } \Omega^-, \\ -\mathrm{div}(A \nabla u^-) + k'(u^-) v^- + v^- = -\mathrm{div}(B \nabla u) & \text{in } \Omega^-, \\ \theta = -\dfrac{1}{\beta} \chi_\omega v^-, \end{cases}$$

together with the boundary conditions

$$\begin{cases} A_0 \nabla_{x''} u^+ \cdot \nu = 0, \quad A_0 \nabla_{x''} v^+ \cdot \nu = B_0 \nabla_{x''} u^+ \cdot \nu \text{ on } \Gamma_a, \\ A \nabla u^- \cdot \nu = 0, \quad A \nabla v^- \cdot \nu = B \nabla u^- \cdot \nu \text{ on } \Gamma_b, \end{cases}$$

and interface conditions on Γ_0

$$\begin{cases} u^+ = u^-, \quad v^+ = v^-, \quad A_0 \nabla_{x''} u^+ \cdot \nu = A \nabla u^- \cdot \nu, \\ (A_0 \nabla_{x''} v^+ - B_0 \nabla_{x''} u^+) \cdot \nu = (A \nabla v^- - B \nabla u^-) \cdot \nu. \end{cases}$$

Corresponding weak formulation is: Given $f \in L^2(\Omega)$ find $(u, v) \in W(\Omega) \times W(\Omega)$ such that

$$\begin{cases} \displaystyle\int_{\Omega^+} A_0 \nabla_{x''} u \nabla_{x''} \psi + |Y(x'')| k(u) \psi + u \psi + \int_{\Omega^-} A \nabla u \nabla \psi + k(u) \psi + u \psi \\ \displaystyle\quad = \int_{\Omega^+} |Y(x'')| (f + \theta) \psi + \int_{\Omega^-} f \psi, \\ \displaystyle\int_{\Omega^+} A_0 \nabla_{x''} v \nabla_{x''} \phi + |Y(x'')| \left(k'(u) v + v \right) \phi + \int_{\Omega^-} A \nabla v \nabla \psi + (k'(u) v + v) \phi \\ \displaystyle\quad = \int_{\Omega^+} B_0 \nabla_{x''} u \nabla_{x''} \psi + \int_{\Omega^-} B \nabla u \nabla \psi, \end{cases} \tag{30}$$

for all $(\psi, \phi) \in W(\Omega) \times W(\Omega)$ with $\theta = -\frac{1}{\beta} \chi_\omega v$.

The next two theorems gives us that the system defined by (30) is the homogenized limit system.

Theorem 8 *Let $(u_\varepsilon, v_\varepsilon)$ and (u, v) be solutions of (29) and (30) respectively. Then as $\varepsilon \to 0$, we have the following strong convergences*

$$\widetilde{u_\varepsilon} - \chi_{\Omega_\varepsilon} u \longrightarrow 0 \ \textit{strongly in } L^2(\Omega),$$

$$\widetilde{\nabla_{x''} u_\varepsilon} - \chi_{\Omega_\varepsilon} \nabla_{x''} u \longrightarrow 0 \text{ strongly in } L^2(\Omega^+)^{n-m},$$
$$\widetilde{\nabla_{x'} u_\varepsilon} - \chi_{\Omega_\varepsilon} \left(-A_1^{-1} A_2\right) \nabla_{x''} u \longrightarrow 0 \text{ strongly in } L^2(\Omega^+)^m,$$
$$u_\varepsilon - u \longrightarrow 0 \text{ strongly in } H^1(\Omega^-).$$

Proof The proof will be the same as we did in last subsection. The only extra term is $\chi_\omega \theta_\varepsilon$. Since ω is compactly contained in Ω^-, and $\|\theta^\varepsilon\|_{H^1(\omega)} \leq C$. Hence, it won't make any issues in any step of the proof we did in the case of homogenization. □

Theorem 9 *Let $(u_\varepsilon, v_\varepsilon)$ and (u, v) be solutions of (29) and (30) respectively. Then we have the following convergences:*

$$\widetilde{v_\varepsilon} \rightharpoonup v \text{ weakly in } L^2(\Omega),$$
$$\widetilde{\nabla_{x''} v_\varepsilon} \rightharpoonup \nabla_{x''} v \text{ weakly in } L^2(\Omega^+)^{n-m},$$
$$\widetilde{\nabla_{x'} v_\varepsilon} \rightharpoonup A_1^{-1} \left(\left(-B_1 A_1^{-1} A_2 + B_2 \right) \nabla_{x''} u - A_2 \nabla_{x''} v \right) \text{ weakly in } L^2(\Omega^+)^m,$$
$$\widetilde{k(v_\varepsilon)} \rightharpoonup |Y(x'')| k(v) \text{ weakly in } L^2(\Omega^+),$$
$$v_\varepsilon \longrightarrow v \text{ weakly in } H^1(\Omega^-).$$

Proof **Step 1: (Convergences)** Since $\|v_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ is bounded, using the properties of unfolding operator defined in Sect. 3.2 we have $\{T^\varepsilon(v_\varepsilon)\}$ is bounded in $L^2((0,1)^m; H^1(\mathcal{G}))$. Also $\{v_\varepsilon\}$ is bounded in $H^1(\Omega^-)$. Hence from weak compactness, there exist $v^+ \in L^2(\Omega_U)$, $v^- \in H^1(\Omega^-)$, $Q_1 \in L^2(\Omega_U)^m$ and $Q_2 \in L^2(\Omega_U)^{n-m}$ such that

$$\begin{aligned} T^\varepsilon(v_\varepsilon) &\rightharpoonup v^+ \text{ weakly in } L^2(\Omega_U), \\ T^\varepsilon(\nabla_{x'} v_\varepsilon) &\rightharpoonup Q_1 \text{ weakly in } L^2(\Omega_U)^m, \\ T^\varepsilon(\nabla_{x''} v_\varepsilon) &\rightharpoonup Q_2 \text{ weakly in } L^2(\Omega_U)^{n-m}, \\ v_\varepsilon &\rightharpoonup v^- \text{ weakly in } H^1(\Omega^-). \end{aligned} \tag{31}$$

From the properties of unfolding, it is easy to see that

$$Q_2 = \nabla_{x''} v^+.$$

Now to identify Q_2 choose ϕ_ε defined in (24) as test function in the variational from (29) to get

$$\int_{\Omega_\varepsilon^+} A \nabla v_\varepsilon \nabla \psi + k'(u_\varepsilon) v_\varepsilon \psi + v_\varepsilon \psi = \int_{\Omega_\varepsilon^+} B \nabla u_\varepsilon \nabla \psi.$$

Apply unfolding and pass to the limit as $\varepsilon \to 0$ using (23) and (31) to get

$$\int_{\Omega_U} (A_1 Q_1 + A_2 Q_2)(\nabla_y \psi)\phi = \int_{\Omega_U} (B_1(-P_1 u + B_2 P_2)(\nabla_y \psi)\phi$$

which implies

$$A_1 Q_1 + A_2 Q_2 = B_1 P_1 + B_2 P_2. \tag{32}$$

Simplify using values of P_1, P_2 and Q_2 to get

$$Q_1 = A_1^{-1}\left(\left(-B_1 A_1^{-1} A_2 + B_2\right)\nabla_{x''} u - A_2 \nabla_{x''} v\right).$$

Then using the averaging property of unfolding operator in (31), we will get the required convergences. Now it is enough to show that (u, v) satisfies (30).Take $\psi \in C^\infty(\bar{\Omega})$ as a test function in the variational form (29), apply unfolding and pass to the limit as $\varepsilon \to 0$ to get

$$\begin{aligned}
&\int_{\Omega_U} A\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\nabla\psi + k'(u)v\psi + v\psi + \int_{\Omega^-} A\nabla u\nabla\psi + k'(u)v\psi + v\psi \\
&\quad = \int_{\Omega_U} B\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}\nabla\psi + \int_{\Omega^-} B\nabla u\nabla\psi.
\end{aligned}$$

Simplify using (32) to get

$$\begin{aligned}
&\int_{\Omega_U} \left(A_3 Q_1 + A_4 Q_2\right)\nabla_{x''}\psi + k'(u)v\psi + v\psi + \int_{\Omega^-} A\nabla u\nabla\psi + k'(u)v\psi + v\psi \\
&\quad = \int_{\Omega_U} \left(B_3 P_1 + B_4 P_2\right)\nabla_{x''}\psi + \int_{\Omega^-} B\nabla u\nabla\psi.
\end{aligned}$$

Simplify using values of P_1, P_2 and Q_2 to get

$$\begin{aligned}
&\int_{\Omega_U} \left(A_4 - A_3 A_1^{-1} A_2\right)\nabla_{x''} v\nabla_{x''}\psi + k'(u)v\psi + v\psi + \int_{\Omega^-} A\nabla u\nabla\psi + k'(u)v\psi + v\psi \\
&\quad = \int_{\Omega_U} \left(-B_3 A_1^{-1} A_2 + A_3 A_1^{-1} B_1 A_1^{-1} A_2 + B_4 - A_3 A_1^{-1} B_2\right)\nabla_{x''} u\nabla_{x''}\psi \\
&\quad\quad + \int_{\Omega^-} B\nabla u\nabla\psi.
\end{aligned}$$

Taking the average using the properties of the unfolding operator, we get

$$\begin{aligned}
&\int_{\Omega^+} A_0\nabla_{x''} v\nabla_{x''}\psi + |Y(x'')|\left(k'(u)v\psi + v\psi\right) + \int_{\Omega^-} A\nabla u\nabla\psi + k'(u)v\psi + v\psi \\
&\quad = \int_{\Omega^+} B_0\nabla_{x''} u\nabla_{x''}\psi + \int_{\Omega^-} B\nabla u\nabla\psi,
\end{aligned}$$

where the coefficients A_0 and B_0 are given by

$$\begin{aligned}
A_0 &= \left|Y(x'')\right|\left(A_4 - A_3 A_1^{-1} A_2\right), \\
B_0 &= \left|Y(x'')\right|\left(-B_3 A_1^{-1} A_2 + A_3 A_1^{-1} B_1 A_1^{-1} A_2 + B_4 - A_3 A_1^{-1} B_2\right).
\end{aligned}$$

Again as in the previous subsection to prove the existence and uniqueness of a solution for the variational form, a major challenge is to show that A_0 and B_0 is coercive. Since we already got a nice form for A_0, we have to find a similar form form for B_0 also. Fortunately, we obtained a matrix expression for B_0 also, which directly implies its coercivity due to the coercivity of A_0 and B_0. Following are the nice matrix expressions for A_0 and B_0

$$A_0 = \left|Y(x'')\right| \left(\left[-A_3 A_1^{-1} \; I\right] A \left[-A_3 A_1^{-1} \; I\right]^t \right) \quad \text{and}$$
$$B_0 = \left|Y(x'')\right| \left(\left[-A_3 A_1^{-1} \; I\right] B \left[-A_3 A_1^{-1} \; I\right]^t \right).$$

By density of $C^\infty(\bar{\Omega})$ in $W(\Omega)$, we v satisfies the limit problem (30) and hence the proof is completed. □

Remark 1 Here we have considered the PDE with the principal part as a divergence form with non-oscillating matrix coefficients. This is only to make the presentation simpler. We can carry out all the results in any finite dimension with more general linear elliptic PDE with principal part as div $\left(A\left(x, \frac{x'}{\varepsilon}\right) \cdot \nabla\right)$ where $A(x, y')$ are uniformly bounded and elliptic $n \times n$ in $\Omega \times Y$ matrices. For this, we have to use the Lemma 7.5, and 7.6, proven in one of our recent articles [45]. As in [45], all the results can be reproduced with cost functional-coefficient as $B\left(x, \frac{x'}{\varepsilon}\right)$ with minor modifications.

Remark 2 In this article, we have focused on applying control away from the oscillating part of the system. There are technical challenges when attempting to apply control directly to the oscillating part, due to the non-linear nature of the system. However, in our previous work on linear equations, we were able to apply control anywhere, including the oscillating part. We are currently working on finding a way to overcome the technical difficulties associated with applying control to the oscillating part in the non-linear case.

4 Conclusion

In conclusion, this article presents a study of the homogenization of optimal control problems governed by semi-linear elliptic PDEs with matrix coefficients in oscillating domains of two different types:
Domain with oscillations in a circular fashion: In the homogenization process, we arrived at a limit problem that is independent of ε. The limit problem consists of derivatives in both x_1 and x_2 directions in such a way that the derivative in the angular direction averages out. In the homogenization of optimal control problems, the coefficient in the limit optimal control problem not only depends on the cost of unhomogenized functional but it is also influenced by the dynamics.
Domains with oscillations in lesser dimensions: In the homogenization process, we arrived at a limit problem that is independent of ε, and the derivative involved in the PDE in x'' direction, where the domain is not oscillating. The derivatives in the oscillating directions x' vanishes from the limit problem. In the homogenization

of optimal control problems, the coefficient in the limit optimal control problem not only depends on the cost of unhomogenized functional but it is also influenced by the dynamics.

The paper involves quite a bit of technicalities due to the presence of the non-linear term. The major issue was the identification of the limit of the non-linear term, where we used the Browder–Minty method, which involves long computations. In general, homogenizing problems in oscillating domains involves lengthy calculations and the non-linear aspect further adds to the complexity. Although the initially considered inhomogenized problems are without any interface conditions, the highly oscillating nature of the boundary led us to limit problems with interface conditions.

Possible directions for future research: We concentrate on implementing control away from the oscillating part of the domain due to the technical complications arising from the non-linear term. It is a fascinating research question to apply control on the oscillating part and perform homogenization with semi-linear PDE. However, this question remains unsolved due to the existing technical difficulties.

Also, in the whole article, we use Hilbert space techniques to analyze because the source term is from L^2 space. It is interesting to do the homogenization problem with L^1 source term, which is currently open.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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